The Impact of the Financial Crisis on the Pricing and Hedging of Interest Rate Swaps

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Abstract

The credit crunch of 2007-2009 implied a change of regime for interest rate swaps. Prior to the crisis, market interest rates showed consistencies that allowed the construction of a single spot curve to price interest rate swaps. In mid-2007 however, several abnormalities could be observed on the interest rate markets which forced market participants to reevaluate their pricing framework. Credit and liquidity issues became highly important in the pricing and hedging of interest rate swaps, and the new market configuration led to the adoption of a multi-curve framework. A single yield curve is no longer adequate to derive discount factors and forward rates as there is now a need for multiple separated curves in order to account for the different dynamics of instruments of different maturities. This thesis describes how the pricing of interest rate swaps has evolved from a single-curve approach to a multi-curve approach. It also compares the two pricing frameworks empirically by hedging an interest rate swap using hedge ratios derived from the different methods. We find that the multiple-curve framework computes higher par swap rates and delta sensitivities and that the single-curve framework underestimates the price of the swap as well as the delta risk that the swap position is exposed to. Hence, the multiple-curve framework should be used when pricing and hedging interest rate swaps today.

Key words: Financial crisis, Credit risk, Interest rate swaps, IRS, FRA, Discount curve, Spot curve, Forward curve, OIS, Basis swaps, Single-curve approach, Multiple-curve approach, Hedging.

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Notations

\( E_t \) Exponential distributed random variable
\( f(t, T) \) Instantaneous forward rate prevailing at time \( t \)
\( F_c(t; T_1, T_2) \) Continuously compounded forward rate at time \( t \) with settlement date \( T_1 \) and maturity \( T_2 \)
\( F_s(t; T_1, T_2) \) Simply compounded forward rate at time \( t \) with settlement date \( T_1 \) and maturity \( T_2 \)
\( h_i \) Hedge ratio for point \( j \)
\( K_{IRS} \) Fixed swap rate (par swap rate)
\( K_{FRA} \) FRA rate
\( K_{OIS} \) OIS rate
\( L(t, T) \) Simply compounded spot rate at time \( t \) with maturity \( T \)
\( m \) Compounding frequency
\( N \) Notional amount
$P(t,T)$ Zero coupon bond price at time $t$ for maturity $T$ (also called discount factor)

$\tilde{P}(t,T)$ Defaultable zero coupon bond price at time $t$ for maturity $T$ with non-zero recovery

$r_s$ Instantaneous rate

$R_{BS}$ Basis swap spread

$R(t,T)$ Continuously compounded spot rate at time $t$ with maturity $T$

$\delta(t,T)$ The year fraction between time $t$ and $T$

$\Delta$ Delta

$\tau$ Default time

$\lambda_t$ Default intensity

$X_t$ Stochastic process

$\phi$ Recovery rate

$\mathcal{F}_t$ Market information available at time $t$

$\mathcal{G}_t$ Information generated by the stochastic process $X_t$ up to time $t$

$\mathcal{H}_t$ Filtration that corresponds to the indicator function $1_{\{\tau \leq t\}}$

$\mathbb{E}[\cdot]$ Expectation under the risk-neutral measure

$\mathbb{E}[\cdot | \mathcal{F}_t]$ Conditional expectation with respect to the market information $\mathcal{F}_t$ under the risk-neutral measure

$\mathbb{P}[\tau > T]$ Survival probability up to time $T$

$\mathbb{P}[\tau < T]$ Default probability up to time $T$
1. Introduction

The market for over-the-counter interest rate derivatives has experienced a tremendous growth in the last decades. Companies and financial institutions frequently use interest rate derivatives for the purpose of managing interest rate risk exposure and the market has grown to contain several instruments and maturities. The most extensively used instrument that accounts for the majority of the interest rate derivatives market, and which is the focus of this thesis, is plain vanilla interest rate swaps.

The pricing of interest rate swaps was long considered straightforward and researchers agreed on the approach to use. In 2007 however, with the onset of the financial crisis, the market condition changed radically and the elementary pricing procedure used among practitioners became unreliable and obsolete. The financial crisis affected the global economy severely and led to liquidity and credit problems for financial institutions and companies worldwide. The market rates that were closely interrelated prior to the crisis became inconsistent and exhibited different liquidity and credit premium (Mercurio, 2009). These credit and liquidity issues were found to have considerable impacts on the prices of financial products, and since the crisis of 2007-2009, one of the main focuses among practitioners is to try to overcome these issues.

The widened basis spreads, i.e. the difference between market rates with different underlying maturities, observed on the interest rate markets following the initiation of the financial crisis in 2007 indicated that the traditional pricing framework had to be revisited. To be able to correctly price interest rate swaps it became necessary to incorporate the credit and liquidity risks of different tenors, i.e. maturities, and consequently, the pricing of swaps has become much more complex in recent years. It is no longer adequate to use only a single yield curve when determining forward rates of different tenors, instead there is now a need for multiple curves. Furthermore, there is also a need for a different discounting practice since the pre-crisis discounting curve is no longer the best risk-free proxy.
1.1. Problem statement

Because of the fact that the interest rate swap market has experienced substantial growth in the last decades and that interest rate swaps have come to be essential risk management tools for practitioners, it is of utmost importance to be able to correctly price this instrument. The financial crisis of 2007-2009 had an immense impact on the pricing of interest rate swaps and the credit and liquidity risks that were considered negligible prior to the crisis turned out to be the main challenges to overcome for practitioners in the post-crisis environment. With this background, it is interesting and highly relevant to examine how the pricing of interest rate swaps has come to change in recent years and what the main differences are between the pre-crisis and post-crisis approaches. Therefore, the main purposes of this thesis are

*to describe the evolvement of interest rate swap pricing and how the financial crisis of 2007-2009 affected the pricing procedure*

and

*to compare the pre-crisis and post-crisis frameworks empirically by pricing and hedging an interest rate swap under both approaches*

The first objective of this thesis is to describe how the procedure for interest rate swap pricing has evolved in recent years. The focus is on how the financial crisis caused a regime change in interest rate markets and how the credit and liquidity issues that arose in mid-2007 forced practitioners to reconsider the traditional pricing approach. The thesis describes how the new market configuration led to the transition from the pre-crisis single-curve approach to the post-crisis multiple-curve approach.

The second objective is to compare the two approaches and demonstrate the main differences between them. This comparison is carried out through the pricing and hedging of an interest rate swap under both frameworks.
1.2. Structure of the thesis

The structure of the thesis is as follows: First in Section 2 we briefly describe the concept of interest rate swaps and the development of the interest rate swap market. Next, in Section 3, we consider the pre-crisis single-curve pricing framework of interest rate swaps. We describe three curves that are important in the pricing procedure; the discount curve, the spot curve, and the forward curve, and we explain how these curves can be constructed through bootstrapping and interpolation. In Section 4 we analyze the events that occurred on the interest rate market during the financial crisis of 2007-2009 and how the crisis brought about a change of regime and a new market environment. Section 5 addresses the post-crisis multiple-curve pricing framework of interest rate swaps and highlights the differences compared to the pre-crisis pricing approach described in Section 3. In Section 6 we consider hedging of interest rate swaps in the pre-crisis and post-crisis environments. Section 7 covers the numerical part of the thesis and compares the two frameworks empirically. Finally, Section 8 concludes the thesis.

2. Interest rate swaps and their market

In this section, we briefly describe interest rate swaps and the evolution of the interest rate swap market. First, in Subsection 2.1, we explain the concept of interest rate swaps and how these contracts are designed. Then, in Subsection 2.2, we describe the rise of the interest rate swap market and how it has developed since the initiation.

2.1. Interest rate swaps

A plain vanilla interest rate swap (IRS), also called fixed-for-floating IRS, is a basic swap agreement between two counterparties, in which fixed and floating periodic interest payments on some notional amount are exchanged. The buyer of the swap is the party that pays the fixed rate and receives the floating rate (fixed rate payer) and the seller of the swap is the party that pays the floating rate and receives the fixed rate (fixed rate receiver). From the perspective of the buyer, the swap is therefore called a payer interest rate swap whereas a swap from the perspective of the seller is called a receiver interest rate swap. In an IRS, as opposed to in a bond, there is no exchange of the notional amount, and the counterparties only exchange the interest differentials, i.e. the difference between the fixed and floating rates. The
fixed rate, paid by the buyer of the swap, is determined at the inception of the swap agreement so that the expected discounted cash flows are equal for the payer and receiver. The fixed rate remains constant until the maturity of the swap, whereas the floating rate, paid by the seller of the swap, is reset periodically and based on a benchmark index interest rate such as Libor or Euribor (explained in more detail in Appendix A).

The cash flows in an interest rate swap is made up of two legs; the fixed leg, which is the stream of cash flows made by the fixed rate payer, and the floating leg, which is the stream of cash flows made by the fixed rate receiver. An interest rate swap denoted in euro usually has a fixed leg with annual payments and a floating leg with semi-annual payments. Thus, the streams of cash flows in a two-year euro interest rate swap is as shown in Figure 2.1, where \( t_0 \) represents the effective date, i.e. the first date when interest begins accruing, and \( T \) represents the maturity of the swap. The IRS in this case thus has two annual calculation periods on the fixed leg and four calculation periods on the floating leg.

![Figure 2.1. Cash flows in a two-year interest rate swap.](image)

When two parties enter into a swap agreement, they agree to exchange future interest rate payments but make no upfront payments. Consequently, the swap contract is designed such that initially, the present value of the floating leg equals the present value of the fixed leg. Discounting the series of future cash flows for each leg is the core when pricing interest rate swaps (or any other fixed-income instrument) since the price of an IRS is the sum of all discounted cash flows.

Three concepts are important when computing the future cash flows in an interest rate swap; the discount factor, the spot rate and the forward rate. The discount factor is used to discount cash flows, the spot rate is the rate that will decide the floating-rate payments and the forward rate is an estimation of the future spot rate. The relationship between the discount factor, forward rate and spot rate is displayed in Figure 2.2. These rates are discussed in more detail in Section 3.
Figure 2.2 Relationship between the discount factor, the forward rate and the spot rate.

Figure 2.2 shows that if the discount factor is known, one can also find the corresponding spot rate and forward rate. Similarly, the discount factor and the spot rate can be derived from the forward rate while the forward rate and the discount factor in turn can be derived from the spot rate. In swap pricing one usually does not refer to discount factors, spot and forward rates for a specific point in time but rather as functions of time illustrated as discount, forward and spot curves.

2.2. Interest rate swap market

There has been a remarkable development in the interest rate swap (IRS) market since its initiation in the early 1980s. The market trades a wide range of maturities and enables market participants to hedge their interest rate risk exposure and to take positions on future pricing movements. Since interest rate swap agreements are traded over-the-counter, i.e. negotiated privately between two counterparties, as opposed to being traded on organized exchanges, the contracts are highly customizable and can be designed such that the specific requirements of both counterparties are met.

The interest rate swap market arose in response to the increased need among financial institutions and corporations to offset their interest rate risks in the late 1970s, when interest rates experienced increasing levels and volatilities. Many corporations and financial institutions had assets and liabilities with different durations, since long-term fixed-rate assets were often financed with short-term floating-rate liabilities, and were therefore seriously affected by increasing interest rates (Bicksler and Chen, 1986). The increased interest rate risk exposure among market participants resulted in a demand for financial products that could reduce the risk exposure, such as interest rate swaps, options and futures.
Since the first swap agreement in 1982, the interest rate swap market has experienced an explosive growth. In the last decade alone, the market has increased from having outstanding notional amounts of approximately $50 trillion in 2001 to over $440 trillion in 2011, which can be seen in Figure 2.3.

![Graph showing the growth of interest rate swap market from 1999 to 2011](image)

*Source: Bank for International Settlements*

**Figure 2.3. Notional amounts outstanding of interest rate swap contracts during 1999-2011 (billions of US dollars).**

The largest and most liquid market is the euro interest rate swap market. In June 2011, the total notional amount outstanding of interest rate swaps was $441,615 billion, of which approximately 40% were traded in euro (EUR) as can be deduced from Table 2.1. The second and third largest currencies for interest rate swap agreements in 2011 were U.S. dollars (USD) and Japanese yen (JPY) with 30% and 13% of notional amounts outstanding, respectively.

<table>
<thead>
<tr>
<th>Interest rate swaps</th>
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<tr>
<td>Total</td>
<td>441,615</td>
</tr>
<tr>
<td>EUR</td>
<td>172,788</td>
</tr>
<tr>
<td>USD</td>
<td>131,959</td>
</tr>
<tr>
<td>JPY</td>
<td>58,188</td>
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*Source: Bank for international settlements*

**Table 2.1. Interest rate swaps by currency - Notional amount outstanding in June 2011 (billions of US dollars).** Euro-denominated swaps dominate the market.
3. Pre-crisis single-curve framework

In this section we will cover the pre-crisis approach for pricing interest rate swaps. First, in Subsections 3.1, 3.2, and 3.3, we explain the concepts of the discount curve, the spot curve, and the forward curve, respectively. Then, in Subsection 3.4, we describe methods for curve construction. Finally, Subsection 3.5 considers the pricing of interest rate swaps in terms of forward rate agreements.

Prior to the financial crisis of 2007-2009, pricing and hedging of interest rate swaps were considered to be straightforward. The pre-crisis standard market practice had been around for almost three decades, since the initiation of the interest rate swap market in the early 1980s, and was based on the building of a single spot curve to calculate forward rates and discount factors. Consequently, the traditional pre-crisis approach for pricing and hedging interest rate swaps has come to be called the single-curve framework.

Ametrano and Bianchetti (2009) summarize the traditional single-curve pricing and hedging framework in the following steps:

1. Select one finite set of vanilla (i.e. basic) interest rate instruments with increasing maturities.
2. Build one spot curve using the selected instruments and bootstrapping (a method for building the curve incrementally in increasing maturity order which will be discussed in Subsection 3.4.1).
3. Compute on the same curve, forward rates and discount factors and work out the prices by summing up the discounted cash flows.
4. Compute the delta sensitivity, i.e. how the prices react to changes in interest rates, and hedge the resulting delta risk using the suggested amounts (hedge ratios) of the same set of vanillas.

In this section we focus on the first three steps, that is, on the construction of a spot curve through bootstrapping and on the computations of forward rates and discount factors to price an interest rate swap. First, however, we make a more exhaustive introduction of the discount, spot and forward curves, which were briefly mentioned in Subsection 2.1. The last step regarding delta hedging is addressed in Section 5.
3.1. The discount curve

The price of any financial instrument, thus also interest rate swaps, is equal to the present value of the financial instrument’s expected cash flows under the so called risk-neutral measure (see e.g. Björk, 2009, p.12). A highly relevant term when it comes to pricing of interest rate swaps is therefore the discount factor. Brigo and Mercurio (2006, p. 3) define a discount factor between time t and T, where \( t < T \), as “the amount at time t that is “equivalent” to one unit of currency payable at time T”. In swap pricing, the price of a zero coupon bond is used as a discount factor. The zero coupon bond guarantees a return of one currency unit at maturity T, without making any coupon payments prior to maturity. At time t, a zero coupon bond with maturity T has a price of \( P(t, T) \) defined as

\[
P(t, T) = \mathbb{E} \left[ \exp \left( - \int_t^T r_s \, ds \right) |\mathcal{F}_t \right]
\]

where \( r_s \) denotes the so called instantaneous rate, \( \mathcal{F}_t \) denotes the market information available at time t and \( \mathbb{E} \) represents the expectation under the risk-neutral probability measure. For a discussion and derivation of Equation (3.1) we refer to Brigo and Mercurio (2006, p. 51). The instantaneous rate is sometimes denoted the short rate and is often modeled as a stochastic process. In a risk-neutral world, all bond prices and derivative prices depend on this short rate process.

A curve of zero coupon bond prices for different maturities is called a discount curve. The discount curve is a powerful tool in interest rate theory since all interest rates can be derived from zero coupon bond prices (Brigo and Mercurio, 2006, p. 6). This relationship will be established in the two forthcoming subsections.

3.2. The spot curve

The spot rate, also called the zero coupon rate or zero yield, is the rate earned on an investment of \( P(t, T) \) units of currency invested at time t with maturity T (Brigo and Mercurio, 2006, p. 6). One distinguishes between continuously and simply compounded spot rates. A simply compounded interest rate compounds m times per year. Following the outline of Filipović (2009, p.8), a simply compounded investment of one currency unit at time t will grow to the amount \( \left( 1 + \frac{L(t, T)}{m} \right)^m \) at time T, where \( L(t, T) \) is the simply compounded interest
rate at time \( t \) with maturity \( T \), and \( m \) is the number of compounding periods per year. As \( m \) tends to infinity, \( m \rightarrow \infty \), the quantity \( \left( 1 + \frac{L(t,T)}{m} \right)^m \) converges to a constant. In view of this we define the continuous compounding interest rate \( R(t,T) \) as

\[
\lim_{m \to \infty} \left( 1 + \frac{L(t,T)}{m} \right)^m = e^{R(t,T) \delta(t,T)} \tag{3.2}
\]

where \( \delta(t,T) \) denotes the year fraction between time \( t \) and \( T \). The definition of the year fraction depends on the day count convention used. The most common day count convention is Actual/360 where one year is considered to be 360 days long. The year fraction, \( \delta(t,T) \) is then defined as

\[
\delta(t,T) = \frac{D_2 - D_1}{360} \tag{3.3}
\]

where \( D_2 - D_1 \) denotes the actual number of days between the dates \( D_1 = \{y_1, m_1, d_1\} \) and \( D_2 = \{y_2, m_2, d_1\} \). Here, \( y \) stands for year, \( m \) for month, and \( d \) for day.

As stated in Subsection 3.1, the spot rate can be expressed in terms of zero coupon bond prices. Since the spot rate at time \( t \) is the constant rate at which the zero coupon bond yields one unit at maturity \( T \), the simply compounded spot rate \( L(t,T) \) can be found by solving the following equation (Brigo and Mercurio, 2006, p. 7)

\[
P(t,T) \left( 1 + L(t,T) \delta(t,T) \right) = 1 \tag{3.4}
\]

which gives

\[
L(t,T) = \frac{1 - P(t,T)}{\delta(t,T) P(t,T)} \tag{3.5}
\]

where, \( P(t,T) \) is the \( T \)-year zero coupon bond price at time \( t \) (also called discount factor), and \( \delta(t,T) \) is the year fraction between time \( t \) and \( T \).

By using Equation (3.2) and the relation \( P(t,T)e^{R(t,T)\delta(t,T)} = 1 \), the simply compounded spot rate \( L(t,T) \) can be transformed into a continuously compounded spot rate \( R(t,T) \) according to

\[
R(t,T) = - \frac{\ln P(t,T)}{\delta(t,T)} \tag{3.6}
\]
The reason why we derive both simple and continuous spot rates is the fact that market interest rates usually are simply compounded rates whereas continuously compounded rates are used when pricing interest rate derivatives (Hull, 2009, p. 77). As is shown later, there is therefore a need to transform market rates to continuous compounding when building the pricing curves.

A curve of spot rates for increasing maturities at a specific date is called a spot curve (also denoted zero coupon curve, yield curve or term structure of interest rates). The most common shape of the spot curve is an increasing function of time. In times of economic downturn, however, the curve can be inverted, which for instance was the case after the crisis in the 1970s when short rates were dramatically increased to tackle the recession.

The spot rate is valid for an investment made at time t. What is the spot rate for the same investment at time t + 1? In the next subsection we will show that a future spot rate can be estimated from the current spot rate.

3.3. The forward curve

A forward rate is an interest rate known at time t, but valid for a future time interval \([T_1, T_2]\) with \(t \leq T_1 \leq T_2\). Here, \(T_1\) is the settlement day and \(T_2\) is the maturity. If the settlement day is equal to t, \(T_1 = t\), the forward rate is simply the spot rate \(R(t, T_2)\) between t and \(T_2\).

According to classical, no-arbitrage interest rate theory, in which there are no risk-free profits, the implied forward rate between time t and \(T_2\) can be derived from two consecutive zero coupon bonds due to the following equality (see e.g. Filipović, 2009, p. 9)

\[
P(t, T_2) = P(t, T_1)P(T_1, T_2). \quad (3.7)
\]

The relationship stated in Equation (3.7) is deeply rooted in classical interest rate theory and states that any cash flow that is discounted from \(T_2\) to t must have the same value regardless of whether it is discounted directly from \(T_2\) to t or whether it is discounted in two steps, from \(T_2\) to \(T_1\) and then from \(T_1\) to t (Bianchetti, 2009). The discount factor \(P(T_1, T_2)\) is called the forward discount factor and is simply the bond price at time \(T_1\) for a zero coupon bond which matures at time \(T_2\).
In Filipović (2009, p.9) the interested reader can find examples of why Equation (3.7) satisfy the no-arbitrage criteria. Recall that an arbitrage opportunity can be defined as (see e.g. Veronesi (2011, p. 7)) a trading strategy that either

1. Has an initial zero cost and generates a certain positive profit in the future, or
2. Generates a positive profit at initiation and has a certain nonnegative payoff in the future.

If the simply compounded spot rate is defined as in Equation (3.5), then Equation (3.7) implies that the simply compounded forward rate $F_s(t; T_1, T_2)$ is defined as

$$F_s(t; T_1, T_2) = \frac{1}{\delta(T_1, T_2)} \left( \frac{1}{p(T_1, T_2)} - 1 \right) = \frac{1}{\delta(T_1, T_2)} \left( \frac{p(t, T_1)}{p(t, T_2)} - 1 \right). \quad (3.8)$$

Similarly, the continuously compounded forward rate $F_c(t; T_1, T_2)$ is implied by the continuously compounded spot rate in Equation (3.6) and is thus defined as

$$F_c(t; T_1, T_2) = -\frac{\ln p(t, T_2) - \ln p(t, T_1)}{\delta(T_1, T_2)}. \quad (3.9)$$

Let $T_1 = T$ in Equation (3.8). If we consider the limit in Equation (3.8) as the maturity $T_2$ approaches the settlement day $T_1 = T$, then the instantaneous forward interest rate prevailing at time $t$, $f(t, T)$, is defined as (see also in Brigo and Mercurio, 2006, p. 12)

$$f(t, T) = \lim_{T_2 \to T^+} F_s(t; T, T_2)$$

$$= -\lim_{T_2 \to T^+} \frac{1}{p(t, T_2)} \frac{p(t, T_2) - p(t, T)}{\delta(t, T_2)}$$

$$= -\frac{1}{p(t, T)} \frac{\partial p(t, T)}{\partial T}$$

$$= -\frac{\partial \ln p(t, T)}{\partial T}. \quad (3.10)$$
This definition implies the following relationship between the zero coupon bond $P(t,T)$ and the instantaneous forward rate $f(t,T)$

$$P(t,T) = \exp\left(-\int_{t}^{T} f(t,u)\,du \right). \quad (3.11)$$

Note that Equation (3.11) confirms that all interest rates can be derived from the zero coupon bond price, which was stated in Subsection 3.1.

Recall Figure 2.2, in which we showed the relationship between the spot rate, the forward rate and the discount factor. From what we have explained in Subsections 3.1, 3.2, and 3.3, it is now possible to elaborate upon Figure 2.2 and include some notations and equations, see Figure 3.1 below.

**Figure 3.1. Interchangeability between discount factors, forward rates and spot rates**

We have now defined the three fundamental curves used in swap pricing and the relationship between them. Next, in Subsection 3.4, we address how to construct these curves in practice.
3.4. Methods for curve construction

Recall that in the single-curve framework, a single curve is used for both discounting and forwarding. By single-curve, it is meant that the same instruments are used to derive all three curves; the discount curve, the spot curve and the forward curve. Since we know that all curves can be derived from each other, there is actually no need to specify exactly which curve is referred to when the term single-curve is used. In literature, however, single-curve often refers to the spot curve, and so we adapt this view.

3.4.1. Bootstrapping

The practical curve building is done using a method called bootstrapping. In the bootstrapping approach, the curve is built incrementally in increasing maturity order. The bootstrapping can be divided into three buckets or segments; the short end, the middle area and the long end of the curve. Which instruments to include in the construction of the spot curve depend on the framework used but in general, deposits, forward rate agreements (FRA), futures and swaps are included. FRAs, which will be discussed in more detail in Subsection 3.5 below, are cash-settled forward contracts where two parties agree to exchange interest rate differentials at the maturity of the contract. The included instruments should be observable, liquid and have similar credit risk (Ron, 2000).

In this thesis, the short end of the spot curve is constructed from interbank deposit rates with maturities ranging from overnight to three months. The two most important interbank rates are Libor (London interbank offered rate) and Euribor (Euro interbank offered rate). See Appendix A for more information on these two reference rates.

Here, we want to construct a curve for the Euro market and consequently Euribor and Eonia, which is the overnight rate, will be used in the short end. Since we know that all interest rates can be derived from the discount factor, the discount factors are bootstrapped from quoted Eonia and Euribor rates using Equation (3.5), which can be rewritten as

\[
P(0,T) = \frac{1}{1 + L(0,T)\delta(0,T)} \tag{3.12}
\]
where \( P(0, T) \) denotes the discount factor, \( L(0, T) \) denotes the simply compounded Euribor market rate for a specific maturity \( T \), and \( \delta(0, T) \) denotes the year fraction between \( t = 0 \) and \( T \). The data used in the bootstrapping process is compiled in Appendix B.

The middle area of the spot curve can be derived from forward rate agreements and/or futures contracts. In this thesis we only use FRAs. Nonetheless, we could just as well have used futures. However, if futures are used, the quoted prices cannot be applied directly but need to be adjusted for convexity. For more details on convexity adjustments, we refer to Kirikos and Novak (1997).

FRAs are quoted as simply compounded forward rates \( F_s(t; T_1, T_2) \) which we prove in Subsection 3.5.1. The corresponding discount factor \( P(0, T_2) \) is bootstrapped using Equation (3.8) so that

\[
P(0, T_2) = \frac{P(0, T_1)}{1 + F_s(0; T_1, T_2)\delta(T_1, T_2)}.
\]

(3.13)

When bootstrapping the middle area of the curve, the previous discount factor \( P(t, T_{1}) \) needs to be known before \( P(t, T_2) \) can be derived. This is exactly what is meant by bootstrapping; the curve is completed by deriving discount factors for different maturities \( T_1 \) in an incremental manner.

The spot curve from two (or three) years onwards is constructed using observed par swap rates, which are quoted swap rates that make the initial value of the swap equal to zero. For very long maturities, only the most liquid swaps are used, which typically are swaps with maturities of 12, 15, 20, 25, and 30 years. Ron (2000) derives the long end continuously compounded spot rates from observed par swap rates by using the following equation

\[
R(0, T) = -\frac{\ln \left[ \frac{1 - \sum_{i=m}^{T-m} \left( K_{IRS} e^{-R(0,T_i)\times\delta(0,T_i)} \right)}{1 + K_{IRS}} \right]}{\delta(0, T)}
\]

(3.14)

where, \( K_{IRS} \) denotes the observed swap rate, explained in more detail in Subsection 3.5 below, and \( R(0, T_1) \) is the continuously compounded spot rate for time \( T_1 \). For instance, to derive the five-year spot rate, the spot rates for the years 1, 2, 3, 4 need to be known (assuming annual payments).
Interpolation

Once the spot rates have been bootstrapped from market data, interpolation, i.e. methods to construct observations between two known points in time, is used to achieve a continuous curve with quotes for all maturities $T_i$. There are several interpolation techniques that can be used in the estimation of the spot curve. The most common methods are based on polynomial splines (Andersen, 2007). Some of the available methods are addressed here.

A “quick and dirty” interpolation method is the linear interpolation where a complete spot curve is constructed using straight lines to connect observed market data points. It is an interpolation process that is simple to implement and that can be defined in closed form. Despite its simplicity it has some serious shortcomings that cannot be overlooked, such as its tendency to generate kinks whenever the slope of the yield curve changes (Ron 2000). A more sophisticated interpolation approach is the piecewise cubic spline interpolation. The use of cubic spline interpolation for estimation of spot curves has become very popular among financial institutions, mainly because of the fact that the first and second derivatives are continuous across all polynomial segments and the curve thus experiences a smoothness constraint (Wolberg, 1999).

The interpolation method used in this thesis is the piecewise cubic spline interpolation. A graph comparing different interpolation methods built-in in Matlab for our data can be found in Appendix C. When we have obtained continuous spot and discount curves, the instantaneous forward rates are found by taking the derivative of the discount factors with respect to time (see Equation (3.10)). The derived curves can now be used to price an interest rate swap. The pricing of interest rate swaps under the single-curve framework is discussed next, in Subsection 3.5.

3.5. Pricing of interest rate swaps in the single-curve framework

At initiation, the value of an interest rate swap (IRS) should be zero, hence, we should find the fixed rate, $K_{IRS}$, that will make the present value of the fixed leg equal to the present value of the floating leg, that is, $PV_{IRS,\text{fixed}} = PV_{IRS,\text{floating}}$. Recall from Subsection 2.1, that the fixed leg represents the stream of cash flows made by the fixed rate payer (buyer of the swap) whereas the floating leg represents the stream of cash flows made by the fixed rate receiver (seller of the swap).
An interest rate swap in the single-curve framework can be priced either in terms of forward rate agreements (FRA), discussed below, or in terms of bonds. No matter the approach used, the fixed payments will always be the same. Here, we focus on how to price an interest rate swap in terms of FRAs. We start by introducing the concept of forward rate agreements in Subsection 3.5.1 and then, in Subsection 3.5.2, we explain how to use FRAs to price an IRS.

### 3.5.1. Forward rate agreements

A forward rate agreement (FRA) is an over-the-counter off-balance sheet derivative instrument, i.e. an instrument that is not recorded on the balance sheet but still generates gains and losses that affect net income. A buyer of an FRA (fixed-rate payer) is long the FRA and borrows a notional amount at an agreed rate of interest, the FRA rate $K_{FRA}$ in exchange for a floating-rate payment. A seller of an FRA (fixed-rate receiver) takes the opposite side by shorting the FRA and thus lending a notional amount for which she receives a fixed-rate payment and pays a floating-rate payment in return. Therefore, a buyer of an FRA gains from increasing interest rates and a seller gains from decreasing interest rates. FRAs are used by financial institutions and large corporations either for hedging or speculative purposes. There is no exchange of the notional amount between the parties trading the FRA, only the difference in interest rates between the FRA rate and the actual rate at the maturity of the FRA, is exchanged. The borrowing or lending is usually done at 3M Euribor or 6M Euribor (or the corresponding Libor). A forward rate agreement is quoted as a 3x6 FRA, a 6x12 FRA, or similar. The first number indicates the months until the FRA expires at time $T_1$ (see Figure 3.2), whereas the second number indicates the months left until the instrument that underlies the FRA matures at time $T_2$ (Kolb and Overdahl, 2007, p.708).

The holder of the forward rate agreement receives an interest rate payment for the contract period between $T_1$ and $T_2$, shown in Figure 3.2. The cash flows are exchanged at maturity $T_2$ but the contract itself is settled at time $T_1$, where $T_1 < T_2$, which means that the expected cash flows must be discounted from $T_2$ to $T_1$.

![Figure 3.2. Illustration of a forward rate agreement](image-url)
At maturity $T_2$, the value of the forward rate agreement for the seller of the FRA (fixed-rate receiver) is

$$N\delta(T_1, T_2)(K_{FRA} - L(T_1, T_2))$$

(3.15)

where $N$ denotes the notional amount in the contract, $\delta(T_1, T_2)$ denotes the year fraction for the contract period $[T_1, T_2]$, $K_{FRA}$ is the FRA rate and $L(T_1, T_2)$ denotes the simply compounded spot rate defined as in Equation (3.5).

By using the definition of the simply compounded spot rate in Equation (3.5) and the relation stated in Equation (3.7), which gives $P(t, T_1) = \frac{P(t, T_2)}{P(T_1, T_2)}$, it is possible to rewrite Equation (3.15) in the following way

$$N\delta(T_1, T_2) \left[K_{FRA} - \frac{1 - P(T_1, T_2)}{\delta(T_1, T_2)P(T_1, T_2)}\right] = N \left[K_{FRA}\delta(T_1, T_2) - \frac{1}{P(T_1, T_2)} + 1\right].$$

(3.16)

To obtain the value of the contract at time $t$, the cash flow exchanged at time $T_2$, shown in Equation (3.16), must be discounted back to time $t$. The contract value at time $t$ is thus defined as

$$NP(t, T_2) \left[K_{FRA}\delta(T_1, T_2) - \frac{1}{P(T_1, T_2)} + 1\right]$$

$$= N(K_{FRA}P(t, T_2)\delta(T_1, T_2) - P(t, T_1) + P(t, T_2)).$$

(3.17)

In order for the forward rate agreement to be fair, the contract value must be zero at time $t$. The FRA rate, $K_{FRA}$, must therefore be set so that Equation (3.17) equals zero. Solving for $K_{FRA}$, we find that the appropriate FRA rate to use in the contract is the simply compounded forward rate, which was defined in Equation (3.8), that is

$$K_{FRA} = \frac{P(t, T_1) - P(t, T_2)}{P(t, T_2)\delta(T_1, T_2)}$$

$$= \frac{1}{\delta(T_1, T_2)} \left(\frac{P(t, T_1)}{P(t, T_2)} - 1\right)$$

$$= F_s(t; T_1, T_2).$$

(3.18)
3.5.2. Pricing an interest rate swap in terms of forward rate agreements

When pricing an interest rate swap (IRS) in terms of forward rate agreements (FRA), the exchanges of interest rate payments can be regarded as FRAs. The first exchange of payments is known at the inception of the contract but the remaining transactions must be computed, which can be done by assuming that forward rates are realized, i.e. that future spot rates equals the forward rates. Hull (2009, p.161) describes the pricing process in the following three steps:

1. Use the spot curve to calculate forward rates for each of the spot rates (Euribor) that will determine swap cash flows.
2. Calculate the swap cash flows by assuming that the future spot Euribor rates will be equal to the forward rates.
3. Discount the swap cash flows using the spot curve to obtain the swap value.

Pricing an interest rate swap basically means determining the terms of the swap at initiation, i.e. determining the fixed rate $K_{IRS}$ that should be exchanged for the floating rate so that the value of the fixed leg equals the value of the floating leg. The fixed swap rate $K_{IRS}$ is determined such that no side of the agreement is given an arbitrage opportunity, which basically means that $K_{IRS}$ is consistent with the spot curve. One way to satisfy this constraint is to use FRA quotations as representatives of the spot curve. Then, the fixed swap rate $K_{IRS}$ is determined by setting the present value of the swap’s fixed leg equal to the present value of a portfolio of FRAs (which represents the swap’s floating leg).

Since the future fixed rate payments are known when entering into the contract at time $t$, the present value of the fixed leg is defined as

$$PV_{IRS,\text{fixed}} = K_{IRS}N \sum_{i=1}^{n} P(t, T_i)\delta(T_{i-1}, T_i)$$  \hspace{1cm} (3.19)

where $K_{IRS}$ denotes the fixed swap rate paid at times $T_i$, where $i = 1, ..., n$. Furthermore, $P(t, T_i)$ denotes the discount factor at time $t$ for maturity $T_i$, defined as in Equation (3.13), $N$ is the notional amount, and $\delta(T_{i-1}, T_i)$ represents the year fraction between times $T_{i-1}$ and $T_i$.

The present value of the fixed leg is thus the sum of the discounted fixed cash flows.

As in the case of the fixed leg, the present value of the floating leg is the sum of the discounted floating rate payments. The floating leg in the IRS pays the Euribor rate with tenor
\([S_{j-1}, S_j]\) at times \(S_j\), where \(j = 1, \ldots, m\). If we assume that forward rates are realized, then we can use FRA quotations to find the present value of the floating leg.

Recall from Equation (3.18) that \(K_{\text{FRA}} = F_s(t; S_{j-1}, S_j)\), i.e. that the fair FRA rate \(K_{\text{FRA}}\) equals the simply compounded forward rate \(F_s(t; S_{j-1}, S_j)\) at time \(t\) for the period \([S_{j-1}, S_j]\), where \(t \leq S_{j-1} \leq S_j\). Hence, the present value of the floating leg is defined as

\[
P_{\text{FRA, floating}} = N \sum_{j=1}^{m} F_s(t; S_{j-1}, S_j) P(t, S_j) \delta(S_{j-1}, S_j) \quad (3.20)
\]

where \(P(t, S_j)\) denotes the discount factor between time \(t\) and time \(S_j\), and \(\delta(S_{j-1}, S_j)\) denotes the year fraction between times \(S_{j-1}\) and \(S_j\).

Given the two increasing dates vectors \(T = \{T_0, \ldots, T_n\}\) and \(S = \{S_0, \ldots, S_m\}\), where \(T_n = S_m > T_0 = S_0 \geq t_0\), the fair swap rate \(K_{\text{IRS}}(t, T, S)\) (denoted \(K_{\text{IRS}}\) for simplicity) can be found by setting Equations (3.19) and (3.20) equal to each other and solving for \(K_{\text{IRS}}\), that is

\[
K_{\text{IRS}} = \frac{\sum_{i=1}^{n} F_s(t, T_i) \delta(T_{i-1}, T_i) = N \sum_{j=1}^{m} F_s(t, S_{j-1}, S_j) P(t, S_j) \delta(S_{j-1}, S_j)}{\sum_{i=1}^{n} P(t, T_i) \delta(T_{i-1}, T_i)}
\]

which gives a fixed swap rate, \(K_{\text{IRS}}\), defined as (see also Bianchetti and Carlicchi, 2011)

\[
K_{\text{IRS}} = \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} F_s(t, S_{j-1}, S_j) P(t, S_j) \delta(S_{j-1}, S_j)}{\sum_{i=1}^{n} P(t, T_i) \delta(T_{i-1}, T_i)}
\]

If we insert the definition of the simply compounded forward rate \(F_s(t; S_{j-1}, S_j)\) (see Equation (3.8)) into Equation (3.21) and use the telescoping sum, i.e. the summation where successive terms cancel each other out, together with the property \(T_n = S_m > T_0 = S_0 \geq t_0\), then the fixed swap rate \(K_{\text{IRS}}\) in Equation (3.21) simplifies to

\[
K_{\text{IRS}} = \frac{\sum_{j=1}^{m} \sum_{i=1}^{n} \left(\frac{P(t, S_{j-1}) - P(t, S_j)}{\delta(S_{j-1}, S_j) P(t, S_j)}\right) P(t, S_j) \delta(S_{j-1}, S_j)}{\sum_{i=1}^{n} P(t, T_i) \delta(T_{i-1}, T_i)}
\]

\[
= \frac{P(t, T_0) - P(t, T_n)}{\sum_{i=1}^{n} P(t, T_i) \delta(T_{i-1}, T_i)}.
\]

(3.22)
4. Market evolution

In this section we address the different abnormalities that could be observed on the interest rate markets from August 2007 and onwards. First, in Subsection 4.1 we describe the divergence between forward rate agreements and implied forward rates. In Subsections 4.2 and 4.3, we describe the increasing basis swap spreads and Euribor-Eonia OIS spreads, respectively. Subsections 4.4 and 4.5 address the issues of credit and liquidity risks, and finally, Subsection 4.6 deals with counterparty credit risk and collateralization.

Prior to the onset of the financial crisis in August 2007, interest rate markets were consistent and rates followed each other closely. There were no significant spreads between Euribor and Eonia overnight index swap (OIS) rates (a fixed swap rate that is paid/received in exchange for a floating rate tied to the Euro overnight rate Eonia), and both rates were assumed to be risk-free. The quotes of forward rate agreements (FRA) were consistent with forward rates implied by two consecutive deposits, so called implied forward rates, and there were no considerable inequalities in the cash flows of interest rate payments with different maturities (Morini, 2009). The pre-crisis theory of derivatives pricing was straightforward and relied on the assumption that borrowing and lending could take place at a unique risk-free rate, as described in Section 3. However, in mid-2007, the market environment changed remarkably, as we explain in more detail in this section.

4.1. Forward rates and forward rate agreements

Prior to 2007, a Forward Rate Agreement (FRA) could be replicated by going long one deposit with the same maturity as the FRA and going short another deposit with a maturity equal to the FRA’s start date (Mercurio, 2009). This feature made it possible to express the exchanges of payments in an interest rate swap in terms of a portfolio of FRAs, as was demonstrated in Subsection 3.5. However, as the crisis progressed this feature became history. The implied forward rates of two consecutive deposits are now different from the quoted FRA rates making them unsuitable for replication.

As discussed in Section 3.3, the forward rate between time t and $T_2$ can be derived from two consecutive deposits due to the classical no-arbitrage relation stated in Equation (3.7)

$$P(t, T_2) = P(t, T_1)P(T_1, T_2).$$

(4.1)
Recall that Equation (3.7) implied that the forward rate $F(t; T_1, T_2)$ can be expressed as

$$F(t; T_1, T_2) = \frac{P(t, T_1) - P(t, T_2)}{\delta(T_1, T_2)P(t, T_2)}. \quad (4.2)$$

In Figure 4.1, Equation (4.2) has been used together with the definition of the zero coupon bond price (or discount factor) defined in Equation (3.12), to calculate implied forward rates from two Eonia swap rates with maturities of three months and six months respectively. The resulting rates are compared to the quotes of a 3x6 Euribor FRA, i.e. a forward rate agreement that settle in three months and matures in six months as explained in Subsection 3.5.1. As clearly can be seen in Figure 4.1, the two rates were replications of each other until mid-2007 when they started to diverge substantially, peaking in fall-2008 at approximately 110 basis points. Even when the financial crisis calmed down, the spread was persistent at around 30 basis points. In Figure 4.1 we also notice the impact of the recent sovereign debt crisis that increased the spread to around 80 basis points in mid-2011.

**Source:** Nordea Analytics/Bloomberg and own computations

**Figure 4.1.** 3x6 forward rates implied by two Eonia swaps compared to Euribor FRA quoted rates. Traditional no-arbitrage relations broke down during the financial crisis of 2007-2009. A forward rate can no longer be implied by two consecutive deposits.
Somewhat surprisingly though, the divergence between FRA and forward rates does not lead to arbitrage opportunities. According to Mercurio (2009), the two rates are in fact allowed to differ and their divergence can be explained by future credit and liquidity issues. This topic is addressed in Subsection 4.5.1 where we show how a simple credit model can justify and explain this divergence between FRA and forward rates.

4.2. Basis swaps

A basis swap (BS) is an interest swap where two floating interest payments are exchanged. The floating-rate is indexed to different tenors (for example, 3M Euribor and 6M Euribor) or different interest rate bases (for example, one leg is tied to the Euribor rate and the other leg to the T-bill rate). The purpose of going into a basis swap is to hedge the interest rate risk that may arise from having liabilities and assets indexed to different floating interest rates or interest rate maturities, i.e. tenors. The risk that the normal relationship between two floating rates might change is denoted basis risk (Kolb and Overdahl, 2007 p.679). In the following, we focus on the risk related to interest rates of different tenors.

Before the crisis, basis swaps could be valued by considering two plain vanilla interest rate swaps where the fixed legs were identical and the floating leg of each swap was indexed to one of the tenors (maturities) in the basis swap. Alternatively, one could think of the basis swap as a floating-for-floating swap with a basis swap spread attached to the floating leg with the shortest tenor (Morini, 2009). Using the latter approach, the present value of a basis swap $PV_{BS}$ is expressed as (see e.g. Bianchetti and Carlicchi, 2011)

$$PV_{BS} = N \left( \sum_{i=1}^{m} P(t,T_i) (F_x(t,T_i) + R_{BS}) \delta(t,T_i) - \sum_{i=1}^{n} P(t,T_i) F_y(t,T_i) \delta(t,T_i) \right) \quad (4.3)$$

where $N$ denotes the notional amount, $P(t,T)$ is the discount factor (the zero coupon bond price), $F_x(t,T_i)$ and $F_y(t,T_i)$ are the forward rates with tenors $x$ and $y$ respectively (e.g. $x = 6$ months and $y = 3$ months), $R_{BS}$ represents the basis swap spread, and $\delta(t,T)$ denotes the year fraction between time $t$ and $T$. The indexes $i$ and $j$ relates to the two legs in the swap.
As in the case of plain vanilla interest rate swaps, the initial value of a basis swap should be zero. Thus, the basis swap spread, $R_{BS}$, is defined as

$$R_{BS} = \frac{\sum_{i=1}^{m} P(t, T_i) F_Y(t, T_i) \delta(t, T_i) - \sum_{i=1}^{m} P(t, T_i) F_X(t, T_i) \delta(t, T_i)}{\sum_{i=1}^{m} P(t, T_i) \delta(t, T_i)}.$$ \hspace{1cm} (4.4)

If we assume that $n=m$ and that the payment frequency of the two legs coincide and payments occur on the same dates such that $i=j$, e.g. both legs are paid semi-annually, then Equation (4.4) can be expressed as

$$R_{BS} = F_Y(t, T_i) - F_X(t, T_i).$$

Thus, the basis swap spread is the difference between the forward rates of two similar interest rates with different maturities. Before the financial crisis of 2007-2009, however, markets were assumed to be risk-free and arbitrage-free and so the interest rate tenor was considered to have no effect on the forward rate (Morini, 2009), which means that the pre-crisis basis swap spread was assumed to be zero, $R_{BS} = F_Y(t, T_i) - F_X(t, T_i) = 0$.

However, in mid-2007 when market conditions changed considerably, basis swap spreads started to increase and became more and more inconsistent. Figure 4.2 shows historical quotes for two different basis swap spreads; one for a swap where 3M Euribor is exchanged for 6M Euribor and another for a swap where 6M Euribor is exchanged for 12M Euribor. As in the case of the spread between FRAs and implied forward rates, discussed in the previous subsection, basis swap spreads reached very high levels in September 2008 when Lehman Brothers filed for bankruptcy. Note, however, that the basis swap spreads have reached even higher levels during the recent sovereign debt crisis. Another interesting observation is the different patterns of the 3M vs. 6M basis swap spread and the 6M vs. 12M basis swap spread. The 3M vs. 6M basis swap spread has constantly increased from mid-2007 until today whereas the 6M vs. 12M basis swap spread shows a much more irregular pattern.
The basis swap spreads increased dramatically in September 2008, after the fall of Lehman Brothers, and have continued to increase with the outburst of the recent sovereign debt crisis. Data from January 1, 2006 until March 9, 2012.

4.3. Euribor-Eonia OIS spread

Apart from the divergences observed between forward rate agreements and implied forward rates (Subsection 4.1), and between basis swaps of different tenors (Subsection 4.2), there is a third important abnormality that can be observed on the interest rate markets since mid-2007. This abnormality is the increased spread between the deposit rate Euribor and the Eonia overnight index swap (OIS) rate, i.e. the fixed swap rate that is paid or received in exchange for a floating rate tied to Eonia. The two rates had followed each other closely prior to the crisis with a spread of only a few basis points. In August 2007 however, the situation changed and the spread between Euribor and Eonia OIS rates increased. Figure 4.3 displays the evolvement of 3M Euribor and 3M Eonia OIS from 2006 until today, clearly showing the structural break that occurred in the autumn of 2007 and how the spread between the two rates increased dramatically.
Figure 4.3. Euribor-Eonia OIS spreads. The two interbank rates were considered to be risk-free and followed each other closely prior to the crisis. The rates are plotted on the left-hand side and the spread is plotted on the right-hand side.

The increased Euribor-Eonia OIS spreads could be observed for all tenors, with the longest tenors showing the largest spreads. Figure 4.4 displays the Euribor-Eonia OIS spreads for the tenors 1M, 2M, 3M and 6M. As can be seen, the spreads were practically zero until mid-2007. Then, they suddenly increased and reached their highest points in September 2008 right after the default of Lehman Brothers. In 2009 and 2010, the spreads decreased but were still much greater than their pre-crisis levels, and as with the other post-crisis abnormalities described in Subsections 4.1 and 4.2, the recent sovereign debt crisis in 2011 caused yet another increase in the spreads. However, at the writing moment (May 2012), the Euribor-Eonia OIS spreads are, once again, displaying lower but not negligible levels.
How can the market abnormalities explained in Subsection 4.1, 4.2 and 4.3 be explained? Researchers and market participants agree that the major explanation relates to credit and liquidity issues. For instance, basis swaps incorporates spot and forward rates that carry credit and liquidity risks. Hence, the basis swap spread outburst can be considered a result of the different credit and liquidity risks of Libor rates with different tenors (Bianchetti and Carlicchi, 2011). The same argument holds for Euribor and Eonia OIS rates and also for forward rate agreements and implied forward rates. When referring to liquidity risks one may mean different definitions. Here, as in Michaud and Upper (2008), we focus on funding liquidity risks, i.e. the risk of not being able to convert assets to cash. The topic of credit and liquidity risks in post-crisis interest rate markets are discussed in more detail in Section 4.4 below.

### 4.4. Credit and liquidity risks

The reason why liquidity and credit risks often are mentioned as one component rather than two is the difficulty of separating these. This is due to the fact that the funding liquidity needs of banks cannot be directly observed (Michaud and Upper, 2008). Nonetheless, Michaud and Upper (2008) states that liquidity needs indeed had an effect on for example the Euribor-Eonia OIS spreads and drove the spreads especially around the turn of the year. Further evidence of this is the differences in Euribor rates across maturities. According to Michaud
and Upper (2008), to solely worry about counterparty risk (credit risk) would have implied much more stable rates across maturities. Filipović and Trolle (2011) also recognize that liquidity matters. They give liquidity hoarding as an example, that is, banks may hoard liquidity in times of distress if they fear that some event could threaten their access to funding. However, due to the difficulties of measuring liquidity risk, the focus of this section and the following subsection is on credit risk. It is nonetheless important for the reader to recognize that liquidity issues also have a part in explaining the spreads observed in mid-2007.

One of the main lessons from the financial crisis of 2007-2009 was that Euribor rates could no longer be treated as risk-free. Suddenly, there was a fear about possible defaults of interbank counterparties. This fear can also be seen in the increased credit default swap (CDS) spreads, i.e. the price of insuring against a default, of Euribor-panel banks, which coincide with the explosion of the Euribor-Eonia OIS spreads, the FRA-implied forward rate spreads, and the basis swap spreads. The CDS spreads of five European banks are shown in Figure 4.5.

![Figure 4.5. CDS spreads of five European banks.](source: Reuters and Bloomberg)

Filipović and Trolle (2011) describe the default risk in the interbank market in terms of different expectations incorporated in the Euribor and Eonia rates, respectively. Present Euribor rates demonstrate the expected future default risk of the banks included in the Euribor panel at that particular time. However, banks can be dropped from the panel if their credit situation considerably worsens. The fixed Eonia rate therefore demonstrates the expected future default risk of the banks included in future Euribor panels. The default risk of future
Euribor banks is lower than the default risk of current Euribor banks (since the panel can be different as banks with better credit quality replace those with worse credit quality). Consequently, the fixed Eonia rate should be lower than the equivalent Euribor rate.

To more explicitly describe the impact of credit issues on the market environment, the concept of intensity-based credit modelling, an important part in credit risk theory, is introduced in the following section, which is based on Section 4 in Herbertsson (2011) (see also Chapter 5 in Lando, 2004).

### 4.5. Intensity-based credit models

In intensity-based credit models the default time of an obligor, denoted \( \tau \), has a default intensity, denoted \( \lambda_t \). Often \( \lambda_t \) is modelled as \( \lambda_t = \lambda(X_t) \) where \( \lambda: \mathbb{R} \mapsto [0, \infty) \) and \( (X_t)_{t \geq 0} \) is a stochastic process that models e.g. the underlying economic environment. The default intensity can be seen as the rate at which the default event is expected to arrive. The default time, \( \tau \), is a random variable and can in the setting of intensity-based credit modelling be defined as “the first time the increasing process \( \int_0^t \lambda(X_s) ds \) reaches the random level \( E_1 \)” (Herbertsson, 2011, p. 38), where, \( E_1 \) is an exponential distributed random variable with parameter 1. This definition is illustrated in Figure 4.6 below and is expressed mathematically as

\[
\tau = \inf \left\{ t \geq 0 : \int_0^t \lambda(X_s) ds \geq E_1 \right\}. \tag{4.5}
\]

![Figure 4.6: Illustration of the default time in an intensity-based credit model.](Source: Herbertsson (2011, p.38))

The default time is the first time the increasing process, modeled in the figure, reaches the random level \( E \).
Let $\mathcal{F}_t$ be the complete market information available at time $t$, that is, everything that can possibly happen lies in this information set. Furthermore, $\mathcal{G}_t^X$ is the information generated by the stochastic process $X_t$ up to time $t$. Finally, there is a third filtration (information set) $\mathcal{H}_t$ that corresponds to the indicator function $1_{(\tau \leq t)}$. This function equals one if there has been a default up to time $t$ and zero otherwise. Formally, $\mathcal{F}_t$, $\mathcal{G}_t^X$, and $\mathcal{H}_t$ are sigma-algebras.

From Equation (4.5), one can prove that (see Herbertsson, 2011, pp.40-42)

$$\lambda(X_t)1_{(\tau > t)} = \lim_{\Delta t \to 0} \frac{\mathbb{P}[\tau \in [t, t + \Delta t)|\mathcal{F}_t]}{\Delta t}$$

(4.6)

and this explains why $\lambda(X_t)$ is denoted intensity for a default time $\tau$ with respect to the information $\mathcal{F}_t$.

Herbertsson (2011, p. 39) states that the conditional survival probability $\mathbb{P}[\tau > T|\mathcal{F}_t]$ at time $T$ given the market information $\mathcal{F}_t$ at $t$ is

$$\mathbb{P}[\tau > T|\mathcal{F}_t] = 1_{(\tau \geq t)} \mathbb{E} \left[ \exp \left( - \int_t^T \lambda(X_s)ds \right) | \mathcal{G}_t^X \right].$$

(4.7)

The probability of surviving beyond $t$ is given by

$$\mathbb{P}[\tau > t] = \mathbb{E} \left[ \exp \left( - \int_0^t \lambda(X_s)ds \right) \right].$$

(4.8)

It is assumed that the risk-free interest rate $r_t$ is a function of $X_t$ and thus that the short rate process $r_t(\omega) = r(X_t(\omega))$ is a stochastic process. Lando (2004, p.116) and Herbertsson (2011, p.43) show that the following equality holds for any function $f(\omega)$

$$\mathbb{E} \left[ \exp \left( - \int_t^T r(X_s)ds \right) f(X_T)1_{(\tau > t)}|\mathcal{F}_t \right]$$

$$= 1_{(\tau > T)} \mathbb{E} \left[ \exp \left( - \int_t^T (r(X_s) + \lambda(X_s))ds \right) f(X_T)|\mathcal{G}_t^X \right].$$

(4.9)

The price at time zero, i.e. today, of a defaultable zero coupon bond with maturity $T$, denoted $\tilde{P}(0, T)$, can now be calculated. Previously in this thesis, the zero coupon bond
(synonymously denoted discount factor) has been considered risk-free. Now, it is assumed that the bond only pays one at maturity T if and only if the issuer has not defaulted before T, that is iff \( \tau > T \). Here, \( \tau \) is the default time for the issuer. If the issuer has defaulted the bondholder gets a recovery, that is, even though there is a default a certain fraction of the investment can be recovered. The recovery is assumed to be a constant between zero and one and is denoted \( \phi \), i.e. \( 0 \leq \phi \leq 1 \). So if e.g. \( \phi = 0.4 \), then the bond holder loses 60% of the face value. If \( \phi = 0 \), everything is lost at the default. If the recovery is paid at maturity, T, the price of the defaulterable zero coupon bond will be

\[
\tilde{P}(0, T) = \mathbb{E} \left[ \exp \left( -\int_0^T r(X_s) \, ds \right) \left( 1_{(\tau > T)} + \phi 1_{(\tau \leq T)} \right) \right]. \tag{4.10}
\]

The first indicator function, \( 1_{(\tau > T)} \), will be one if \( \tau > T \), namely if the issuer has survived until maturity T, whereas the second indicator function, \( 1_{(\tau \leq T)} \), will be one if the issuer default before time T. Consequently, the sum of the two will be one for certain (since it is impossible for the firm to default and survive at the same time). Given this, Equation (4.10) then simplifies to

\[
\tilde{P}(0, T) = \mathbb{E} \left[ \exp \left( -\int_0^T r(X_s) \, ds \right) \left( \phi + (1 - \phi)1_{(\tau > T)} \right) \right]. \tag{4.11}
\]

If the interest rate is independent of the default intensity, and consequently also the default time, the expectations can be separated and written as

\[
\tilde{P}(0, T) = \mathbb{E} \left[ \exp \left( -\int_0^T r(X_s) \, ds \right) \right] \mathbb{E} \left[ \phi + (1 - \phi)1_{(\tau > T)} \right]. \tag{4.12}
\]

Since the recovery is a constant and since the expectation of an indicator function is the probability of that event occurring, i.e. \( \mathbb{E} [1_{(\tau > T)}] = \mathbb{P}[\tau > T] \), then Equation (4.12) can be simplified to

\[
\tilde{P}(0, T) = P(0, T) \left[ \phi + (1 - \phi)\mathbb{P}[\tau > T] \right]. \tag{4.13}
\]
4.5.1. Using intensity-based credit modeling to explain post-crisis abnormalities

Mercurio (2009) uses the above reasoning to explain the abnormal post-crisis spreads observed in the market (recall Subsections 4.1, 4.2, and 4.3) with a simple credit model. Consider the default time of a generic interbank counterparty to be a random variable denoted by $\tau$. The recovery rate, which is assumed to be constant, is denoted by $\phi$. Furthermore, default and interest rates are assumed to be independent. The initial value of a deposit, i.e., money placed on a bank account with interest, starting at time $t$ with maturity $T$ can be defined as

$$ \bar{P}(t, T) = \mathbb{E}\left[ \exp\left( - \int_t^T r(u) \, du \right) \left( \phi + (1 - \phi)1_{(\tau > T)} \right) \bigg| \mathcal{F}_t \right] $$

$$ = P(t, T) \left( \phi + (1 - \phi)\mathbb{E}\left[1_{(\tau > T)} \big| \mathcal{F}_t \right] \right) \quad (4.14) $$

where $r$ is the default-free instantaneous interest rate, $\mathbb{E}$ is the expectation under the risk-neutral measure, $P(t, T)$ is the price of a default-free zero coupon bond at time $t$ with maturity $T$ and $\mathcal{F}_t$ is the market information available at time $t$. Hence, the deposit described in the paper of Mercurio (2009) is the price of the defaultable zero coupon bond $\bar{P}(t, T)$, with non-zero recovery, derived in the previous section and stated for time $t = 0$ in Equation (4.13).

Mercurio (2009) denotes $\mathbb{E}\left[1_{(\tau > T)} \big| \mathcal{F}_t \right]$ by $Q(t, T)$ for notional convenience. As has previously been explained, the expectation of an indicator function is the probability of the condition being true, that is, $\mathbb{E}\left[1_{(\tau > T)} \big| \mathcal{F}_t \right] = \mathbb{P}[\tau > T | \mathcal{F}_t]$ where $T \geq t$. Recall that the Libor rate $L(T_1, T_2)$, which is the rate earned by the deposit $\bar{P}(T_1, T_2)$ was defined in Section 3 as

$$ L(T_1, T_2) = \frac{1}{\delta(T_1, T_2)} \left[ \frac{1}{\bar{P}(T_1, T_2)} - 1 \right]. \quad (4.15) $$

Substituting for the price of the risky deposit $\bar{P}(t, T)$ in Equation (4.15) gives

$$ L(T_1, T_2) = \frac{1}{\delta(T_1, T_2)} \left[ \frac{1}{P(T_1, T_2)} \left( \phi + (1 - \phi)Q(T_1, T_2) \right) - 1 \right]. \quad (4.16) $$
Mercurio (2009) applies his model to the spread between forward rate agreements (FRA) and implied forward rates, which was addressed in Subsection 4.1. Recall that, prior to 2007, an FRA could be replicated by going long one deposit with the same maturity as the FRA, i.e. $T_2$, and going short another deposit with a maturity equal to the FRAs settlement date, $T_1$. This replication is no longer suitable since the FRA rate $K_{FRA}$ and the forward rate $F_D$, implied by the two consecutive deposits with maturities $T_1$ and $T_2$, are now different. We explain this by following the outline of Mercurio (2009).

Suppose that market participants implement a strategy where they

1) Enter a payer FRA with maturity $T_2$, i.e. take a long position in the FRA and pay the fixed rate $K_{FRA}$, paying out an amount equal to

$$\frac{\delta(T_1, T_2)(L(T_1, T_2) - K_{FRA})}{1 + \delta(T_1, T_2)L(T_1, T_2)}$$

(4.17)

at time $T_1$, where $L(T_1, T_2)$ denotes the Libor rate (or Euribor) at time $T_1$ with maturity $T_2$ defined in Equation (4.15) and $\delta(T_1, T_2)$ represents the year fraction between $T_1$ and $T_2$.

2) Buy $(1 + \delta(T_1, T_2)F_D)$ deposits with maturity $T_2$, thus paying an amount of

$$(1 + \delta(T_1, T_2)F_D)\tilde{P}(t, T_2) = \tilde{P}(t, T_1)$$

where $\tilde{P}(t, T)$ represents the price of the deposit at time $t$ for maturity $T$ and $F_D$ is the implied forward rate.

3) Sell one deposit with maturity $T_1$ and receive an amount equal to $\tilde{P}(t, T_1)$.

The current value of this strategy is zero. At time $T_1$, 1) and 3) above, i.e. the payer FRA and the deposit sold, yields

$$\frac{\delta(T_1, T_2)(L(T_1, T_2) - K_{FRA})}{1 + \delta(T_1, T_2)L(T_1, T_2)} - 1 = -\frac{1 + \delta(T_1, T_2)K_{FRA}}{1 + \delta(T_1, T_2)L(T_1, T_2)}$$

(4.18)

which is negative if rates are assumed to be positive. The residual debt is then paid by selling 2), i.e. the $1 + \delta(T_1, T_2)F_D$ deposits with maturity $T_2$. 

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If we assume that the forward rate agreement contain no counterparty risk, i.e. there is no risk that a counterparty defaults, then by using Equation (4.18) and the fact that the current value is zero as explained above, the value of the FRA at time zero is found by discounting the payoff at time $T_1$

\[
0 = E\left[ P(0, T_1) \frac{\delta(T_1, T_2) (L(T_1, T_2) - K_{\text{FRA}})}{1 + \delta(T_1, T_2) L(T_1, T_2)} \right]
\]

\[
= E\left[ P(0, T_1) \left(1 - \frac{1 + \delta(T_1, T_2) K_{\text{FRA}}}{1 + \delta(T_1, T_2) L(T_1, T_2)} \right) \right]
\]

\[
= E\left[ P(0, T_1)(1 - (1 + \delta(T_1, T_2) K_{\text{FRA}}) P(T_1, T_2)(\phi + (1 - \phi)Q(T_1, T_2))) \right]
\]

which gives

\[
0 = P(0, T_1) - (1 + \delta(T_1, T_2) K_{\text{FRA}}) P(0, T_2)(\phi + (1 - \phi)E[Q(T_1, T_2)])
\]

see also p.8 in Mercurio (2009). From Equation (4.19) it is then possible to define the FRA rate $\tilde{K}_{\text{FRA}}$ as

\[
\tilde{K}_{\text{FRA}} = \frac{1}{\delta(T_1, T_2)} \frac{P(0, T_1)}{P(0, T_2)} \frac{1}{\phi + (1 - \phi)E[Q(T_1, T_2)] - 1}.
\]

(4.20)

Note that this FRA rate is not the same as the FRA rate in Subsection 3.5.

Furthermore, since the recovery rate is a fraction between zero and one, that is $0 \leq \phi \leq 1$ and since $0 < Q(T_1, T_2) < 1$ we get that

\[
0 < \phi + (1 - \phi)E[Q(T_1, T_2)] < 1
\]

and this observation in Equation (4.20) together with Equation (3.18) implies that

\[
\tilde{K}_{\text{FRA}} > \frac{1}{\delta(T_1, T_2)} \frac{P(0, T_1)}{P(0, T_2)} - 1.
\]

(4.21)
Hence, from Equation (4.21) we see that the FRA rate $\tilde{K}_{\text{FRA}}$ is greater than the forward rate $F_D$ implied by the two default-free bonds $P(0, T_1)$ and $P(0, T_2)$.

### 4.6. Counterparty credit risk and collateralization

During the financial crisis it became evident that financial institutions were more interrelated than one thought and that the default of one financial entity could lead to the default of others. Capponi (2011) argues that counterparty credit risk was one of the major drivers of the financial crisis starting in 2007. As has previously been addressed, the existence of counterparty risk can also explain some of the abnormalities seen in the interest markets from 2007 and onwards.

Counterparty credit risk can be defined as the risk that a counterparty will default and consequently not be able to honor its contractual obligations. It is a bilateral risk of loss where both parties are sensitive to the default of the other party. Counterparty risk is present in all financial over-the-counter (OTC) contracts and solely arises from OTC derivatives since exchange-traded derivative contracts are cleared by the exchange thus removing the counterparty risk (Bielecki et al., 2011). Collateralization means that the counterparties regularly post collateral based on some predetermined rules, typically when the NPV (net present value of the contract) breaches, some specific thresholds, or when the CDS rate (or credit rating) of some counterparty deteriorates.

The International Swaps and Derivatives Association (ISDA) mention a number of ways to reduce the credit risk that arises from derivatives transactions. For example to include close-out netting by holding capital against the exposure or to reimburse losses through financial guarantees by another counterpart. However, the most extensively used method for mitigating counterparty credit risk of OTC derivatives positions is collateralization.

In 2010, 70 percent of all OTC derivatives transactions were subject to collateral agreements, and if only the largest derivatives dealers are taken into account this figure is even higher. Of all the OTC derivatives transaction executed by the fourteen firms that represents the world’s
largest derivatives dealers (the so called G14 derivative dealers)\(^1\), around 80 percent were collateralized in 2010, as can be seen in Table 4.1.

<table>
<thead>
<tr>
<th></th>
<th>All (%)</th>
<th>Large dealers (%)</th>
<th>Medium/Small dealers (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>All OTC derivatives</td>
<td>69.8</td>
<td>80.2</td>
<td>66.3</td>
</tr>
<tr>
<td>Fixed Income derivatives</td>
<td>78.6</td>
<td>87.9</td>
<td>74.6</td>
</tr>
<tr>
<td>Credit derivatives</td>
<td>93.2</td>
<td>95.8</td>
<td>91.7</td>
</tr>
<tr>
<td>FX derivatives</td>
<td>58.2</td>
<td>65.2</td>
<td>53.0</td>
</tr>
<tr>
<td>Equity derivatives</td>
<td>72.1</td>
<td>73.2</td>
<td>71.5</td>
</tr>
<tr>
<td>Commodities*</td>
<td>59.6</td>
<td>62.9</td>
<td>56.7</td>
</tr>
</tbody>
</table>

*Including precious metals

Source: International Swaps and Derivatives Association

Table 4.1. Trades subject to collateralization in 2010 (average figures).

5. Post-crisis multiple-curve framework

In this section, we address the post-crisis approach for pricing interest rate swaps. First, in Subsection 5.1, we describe the importance of the discount curve. Then, in Subsection 5.2, we cover the construction of forward curves and stress the importance of homogeneity in the underlying rate tenors. Finally, Subsection 5.3 considers the post-crisis pricing of interest rate swaps.

As we have seen in the previous section, the interest rate market condition changed dramatically in the second half of 2007 and several anomalies could be observed. Primary interbank interest rates such as Euribor and Eonia, for instance, started to diverge strongly. From being stable at a few basis points, the spreads became more and more inconsistent and peaked at 200 basis points in late 2008 (Figure 4.4). There were also divergences among other rates which used to be consistent prior to the crisis, such as among swap rates with different floating leg tenors (basis swaps) and between forward rate agreements and implied forward rates (Bianchetti and Carlicchi, 2011). Credit and liquidity issues, which were previously ignored, became highly relevant forcing risk managers and market operators to rethink the

\(^1\) The world’s largest derivatives dealers are Bank of America Merrill Lynch, Barclays, BNP Paribas, Citigroup, Credit Suisse, Deutsche Bank, Goldman Sachs, HSBC, JP Morgan Chase, Morgan Stanley, Societe Generale, The Royal Bank of Scotland, UBS, and Wells Fargo (ISDA 2011)
long existing pricing principles of interest rate instruments. The pricing approach described in
Section 3 is no longer reliable and has been abandoned. Instead, practitioners now use a so
called multiple-curve approach.

As in the single-curve framework, the curves in the multiple-curve framework have three
different forms; the discount curve (curve of zero coupon bond prices), the spot curve, and the
forward curve (the instantaneous forward rate curve). However, instead of having one curve
of each form, the multiple-curve approach has different curves depending on the underlying
tenor of the bootstrapping instruments. The most common tenors are 1M, 3M, 6M and 12M,
which means that it is common to construct four different spot and forward curves. These
curves, however, are not used for discounting. Instead a discount curve is constructed
separately, with different bootstrapping instruments than those in the construction of the
forward curves.

According to Ametrano and Bianchetti (2009), the multiple-curve framework for pricing and
hedging interest rate derivatives (hence, interest rate swaps) can be summarized as follows:

1. Construct a single discounting curve.
2. Select different sets of vanilla interest rate instruments, each with homogenous
   underlying rate tenors (e.g. 1M, 3M, 6M, 12M).
3. Construct multiple separated forwarding curves, one for each tenor, using the selected
   instruments and bootstrapping.
4. Compute forward rates on each forwarding curve as well as the cash flows relevant for
   pricing derivatives on the same underlying.
5. Compute discount factor using the discounting curve and sum up the discounted cash
   flows in order to work out prices.
6. Compute the delta sensitivity and hedge the delta risk with the appropriate hedge ratio
   of the corresponding set of vanilla instruments.

In this section, we focus on step one to five whereas the hedging is discussed in Section 6. We
start with the discount curve in Subsection 5.1, move on to the forwarding curve in Subsection
5.2, and finally, in Subsection 5.3 we cover how interest rate swaps are priced in the multiple-
curve framework.
5.1. OIS discounting

Prior to the crisis of 2007-2009, the discount curve was derived from the same set of bootstrapping instruments as the forward and spot curves. This curve is usually denoted the Euribor discount curve since the bootstrapping instruments consist of the most liquid Euro-denominated interest rate instruments for different maturities. The Euribor discount curve has long been considered risk-free but, as explained in Section 4, the market environment has radically changed in recent years and rates have increased as a consequence of credit risk premiums. Hence, Euribor is no longer the best risk-free proxy and market participants have begun to move away from traditional discounting practice.

Instead, Eonia is considered the truly risk-free rate, which has led to the adoption of a discount curve constructed of Eonia overnight index swap rates. In an overnight indexed swap (OIS), a fixed-rate cash flow is exchanged for a floating-rate cash flow indexed to a compounded overnight rate, which in the Euro area is the Eonia rate. If the OIS is indexed to Eonia, the swap is called an Eonia swap. By replacing the Euribor curve with an OIS curve when pricing (collateralized) swap contracts, the credit and liquidity risks that are priced into Euribor are eliminated (Smith 2012).

The OIS markets have become increasingly liquid in recent years and the maturities of overnight index swaps have been extended. In the Euro market, Eonia swaps are now available out to 30 years making it possible to construct a complete risk free curve.

In this thesis we construct the OIS discount curve by bootstrapping discount factors from Eonia swaps with maturities ranging from overnight to 30 years. The complete curve is then found by interpolating between quotes using the same interpolation scheme as in the single-curve framework (see Subsection 4.2.1).

5.2. The forward curve in the multiple-curve framework

Recall the road map to multiple-curve pricing of Ametrano and Bianchetti (2011) presented in the beginning of Section 5. The second, third and fourth steps in this guide relates to the construction of the forward curve(s). The building of the forward curve in the multiple-curve framework resembles a lot of the building of the unique spot curve in the single-curve approach. Several instruments with different maturities are chosen. However, instead of mixing different underlying interest rate tenors, the instruments in the multiple-curve
approach share the same underlying tenor as the instrument that is to be priced. This criterion is often stated as a requirement of homogeneity (Bianchetti, 2009).

Consequently, if we would want to hedge a portfolio of interest rate derivatives with different underlying tenors, we would need several forwarding curves. For example, a portfolio consisting of swaps and futures requires one forwarding curve built of instruments with six-month Euribor as underlying (since the floating leg in Euro swaps is paid semi-annually) plus one forwarding curve built of instruments with three-month Euribor as underlying (since cash flows in futures contracts are paid quarterly). This is why the new pricing method is called multiple-curve framework even though in some cases, only two curves are required (if the portfolio consists of a single instrument).

Since longer tenors are considered to be more risky, forwards related to longer tenors should also be priced higher than forwards with shorter tenors (Pallavicini and Tarenghi, 2010). This difference is shown in Figure 5.1, in which we illustrate a hypothetical set of forward curves with different underlying tenors.

![Figure 5.1. Illustration of forward curves with different underlying rate tenors.](image)

The longer the tenor the riskier, and consequently investors demand a higher rate of interest.

In the post-crisis bootstrapping of the forwarding curve, we follow the approach of Bianchetti and Ametrano (2009). They construct the forward curve in roughly the same way as in the single-curve framework, that is, discount factors are bootstrapped from market quotes and then transformed to spot and forward rates. The only difference is that the bootstrapping
instruments are homogenous in the underlying tenor, that is, they have the same tenor. Other methods for building the forward curve are suggested in Henrard (2009) and to some extent also in Chibane and Sheldon (2009) who use the OIS discount factors also in the calculation of the forward rates. This second approach will especially have implication on the hedging of interest rate swaps, which we show in Section 6.

Suppose we want to derive a forwarding curve for the six-month-tenor. To satisfy the requirement of homogeneity we then need to change the set of bootstrapping instruments used in the single-curve bootstrapping process (explained in Subsection 3.4.1). This needs to be done only for the short end and middle area of the curve since the long end is comprised of swaps which already have an underlying of six months.

The forward rate agreement (FRA) has become increasingly popular after the crisis, since it is the only instrument that can provide information for different tenors when constructing the middle area of the curve (Henrard, 2009). There are FRAs quoted on the market with Euribor tenors of three, six and twelve months whereas futures (which can also be used in the middle area) are only available with tenors of three months. In this thesis, the middle area is constructed using FRAs indexed to 6M Euribor.

In contrary to the classical single-curve framework, we do not have any observations in the short end since the criteria to qualify as a bootstrapping instrument for the forwarding curve requires the instrument to have the 6M Euribor as underlying (homogeneity requirement). Consequently, the shortest maturity is six months. However, to avoid negative forward rates in the interpolation process, the overnight rate and the 3M Euribor is added to the short end, as suggested in Henrard (2009). See Appendices B and D for the data used in the construction of the curve.

5.3. Pricing in the multiple-curve framework

The need for two (or more) curves has led to a revision of the pre-crisis pricing procedure for interest rate derivatives. Today, there is a need for a unique discount curve, and since Libor (and also Euribor) can no longer be considered risk-free, overnight indexed swap (OIS) rates \( K_{OIS} \) which are considered the best risk-free proxy, must be used for building the discount curve.
From this discount curve we can derive the multiple-curve risk-free discount factors $P_{OIS}(t, T)$ defined as

$$P_{OIS}(t, T) = \frac{1}{1 + K_{OIS}(t, T)\delta(t, T)} \quad (5.1)$$

where $K_{OIS}(t, T)$ denotes the OIS rate (Eonia OIS rate in the Euro area) at time $t$ with maturity $T$, and $\delta(t, T)$ represents the year fraction between time $t$ and $T$.

The multiple-curve approach also requires the construction of multiple distinct forwarding curves, each homogenous in their underlying rate tenors. To have instruments with homogenous underlying tenors is of utmost important since similar market instruments with different tenors, e.g. forward rate agreements (FRA) on 3M Euribor and interest rate swaps (IRS) on 6M Euribor, may display very different price levels. Thus, the multiple-curve framework also requires a second discount factor, which is related to the forward curve. Contrary to the OIS-discount factor $P_{OIS}(t, T)$, this second discount factor still uses the Libor (or Euribor) rate as the risk-free rate and will therefore be denoted $P^x_L(T_1, T_2)$ where the superscript $x$ demonstrates the homogeneity requirement and will take the values of the underlying tenors, e.g. $x = \{1M, 3M, 6M, 12M\}$. The second discount factor $P^x_L(T_1, T_2)$ in the multiple-curve framework can thus be defined as

$$P^x_L(T_1, T_2) = \frac{1}{1 + L^x(T_1, T_2)\delta^x(T_1, T_2)} \quad (5.2)$$

where $L^x(T_1, T_2)$ is the risky spot Libor (Euribor) rate at time $T_1$ for maturity $T_2$.

Next, for each interest rate coupon, the FRA rate $K^x_{FRA}$ with tenor $x$ is computed using the corresponding forward curve and applying the following formula

$$K^x_{FRA} = F^x(t; T_1, T_2) = \frac{1}{\delta^x(T_1, T_2)} \left( \frac{P^x_L(t, T_1)}{P^x_L(t, T_2)} - 1 \right) \quad (5.3)$$

where $\delta^x(T_1, T_2)$ denotes the year fraction between time $T_1$ and $T_2$.

Given the two increasing date vectors $T = \{T_0, \ldots, T_n\}$ and $S = \{S_0, \ldots, S_m\}$, where $T_n = S_m > T_0 = S_0 \geq t_0$, the fixed swap rate $K^x_{IRS}(t; T, S)$ (denoted $K^x_{IRS}$ for simplicity) in the multiple-curve framework for the underlying tenor $x$, is thus defined as (see also Ametrano and Bianchetti, 2009)
\[ K_{\text{IRS}}^x = \frac{\sum_{j=1}^{m} K_{\text{FRA}}^x P_{\text{OIS}}(t, S_j) \delta(S_{j-1}, S_j)}{\sum_{i=1}^{n} P_{\text{OIS}}(t, T_i) \delta(T_{i-1}, T_i)}. \] (5.4)

Note that Equations (5.2), (5.3) and (5.4) are just generalizations of the corresponding single-curve discount factor, FRA rate and swap rate given by Equations (3.12), (3.18) and (3.21).

6. Hedging of interest rate swaps

In this section we address the hedging of interest rate swaps. First, in Subsection 6.1 we introduce some useful terms and general concepts regarding swap risk management. Then, in Subsections 6.2 and 6.3 we describe delta-hedging under the single-curve and multiple-curve frameworks respectively.

6.1. General concepts and useful terms

An interest rate swap (IRS) position is exposed to a considerable amount of interest rate risk. For a receiver IRS, that is, a swap that pays floating and receives fixed, the risk is increasing interest rates. If the holder of the receiver swap wishes to close the position by going into an opposite position with the same notional, i.e. by entering a payer interest rate swap, she will make a loss if interest rates have increased since the day of initiation. Because the floating leg in the two swaps will be the same, the loss is the difference in the fixed rates for the remaining fixed-rate payments. If the fixed-rate is paid annually, the loss will therefore be \(N[K_{\text{IRS}}(t, T_{i-1}) - K_{\text{IRS}}(t, T_i)]\) per remaining year, where \(N\) denotes the notional amount, \(K_{\text{IRS}}(t, T_{i-1})\) represents the initial fixed swap rate and \(K_{\text{IRS}}(t, T_i)\) is the fixed swap rate today, i.e. the date when the position is to be closed. Similarly, a payer interest rate swap, that is, a swap that receives floating-rate payments and pays fixed, is exposed to the risk of decreasing interest rates.

The loss of a ten-year receiver interest swap that is closed four years into the contract period, after an increase in interest rates, is illustrated in Figure 6.1. If the swap is a payer IRS, the patterned area is instead a profit.
Figure 6.1. Loss of a ten-year receiver interest rate swap that is closed after an increase in interest rates. The loss is the difference between the original swap rate and the swap rate four years after initiation times the remaining time to maturity. If the swap is a payer interest rate swap the patterned area is instead a profit.

The “Greeks” is a well-known term when it comes to the hedging of derivatives. The most common Greeks are Delta, Gamma, Theta, Vega, and Rho (in Greek letters, $\Delta$, $\gamma$, $\theta$, $V$, and $\rho$, respectively) where each letter measures different types of sensitivities. The delta is probably the most well-known hedging parameter and is defined as the rate of change of the value of the derivative, or a portfolio of derivatives, with respect to the underlying asset. For notional convenience, let $L_i$ denote the market rate $L(t,T_i)$ where we have suppressed the dependence of $t$. Since the underlying asset of an interest rate derivative is the interest rate itself, the delta for interest derivatives, such as interest rate swaps, is then defined as

$$\Delta = \frac{\partial PV}{\partial L_i}$$  \hspace{1cm} (6.1)

where $PV$ denotes the value of the derivative.

Recall that an interest rate swap is not valued by using a single interest rate but by using curves of interest rates for different maturities. Therefore, the delta risk of an interest rate swap is often thought of as the risk related to a shift in the spot curve. Since the spot curve can change in many ways the calculation of the delta is not straightforward and can differ from one trader to another. Hull (2009, p. 663) lists four computation alternatives:
1. Calculate the DV01, that is, the impact of a one-basis-point parallel shift in the spot curve.
2. Calculate the impact of a small change in each quote of the bootstrapping instruments.
3. Divide the spot curve into buckets (sections) and calculate the impact of a one-basis-point shift in each bucket.
4. Conduct a principal component analysis.

According to Benhamou and Nodelman (2005), the second approach is called the full delta and is the measure preferred by practitioners. The full delta is the most exact delta since it isolates the effect of each market instrument and can also be used to calculate other types of delta risk.

By using alternative number two, the full delta, it is possible to find out to which interest rates the swap, or swap portfolio, is most vulnerable. Miron and Swannell (1991, p. 133) define a vector of such vulnerabilities as a delta vector. More explicitly, the delta vector is found by changing one of the bootstrapping instruments by one basis point (0.0001) and then comparing the original present value of the swap, prior to the change, with the new present value of the swap, after the change. The change is the delta associated with that specific pillar (point) of the curve, for example the 3-month point, the 5-year point, etc. According to Miron and Swannell (1991, p. 137), the sum of two or more deltas in the delta vector corresponds to the effect of a simultaneous change in those specific interest rates. This is due to the fact that the present value of a swap is a linear function of input rates. Consequently, the sum of all the elements in the delta vector corresponds to a parallel shift in the spot curve.

Miron and Swannell (1991, p. 138) also state that the elements in the delta vector will depend on the framework that is used to value the interest rate swap or portfolio (since the delta is a difference between to present values). Thus, it is interesting to examine how the hedging differs between the pre-crisis single-curve framework and the post-crisis multiple-curve framework.

6.2. Hedging in the single-curve framework

In the single-curve framework, the same curve is used to derive both forward rates and discount factors. Hence, the delta risk is the risk associated with a shift in that single curve. The delta vector is derived, as described above, by “bumping” the quote of each bootstrapping
instrument used to construct the single spot curve by one basis point and thereafter measure the change in the present value. The total delta risk is the total profit or loss from a simultaneous one-basis-point-change in all bootstrapping instruments. Once the risk has been localized the hedging process can begin.

For that purpose, the equivalent position, introduced in Miron and Swannell (1991, p. 133), is useful. By the equivalent position, the authors mean that an interest rate risk exposure can be offset by taking positions equivalent to the different deltas in the delta vector. For example, following Miron and Swannell (1991, p.139), the five-year equivalent position for a swap portfolio is the amount of a par five-year swap that has the same five-year delta as the swap portfolio. In other words, the equivalent position can be expressed as

\[
\text{Equivalent position} = N \frac{\Delta_{5Y}^\pi}{\Delta_{5Y}^{\text{par}}}
\]  

(6.2)

where, \(N\) is the notional amount, \(\Delta_{5Y}^\pi\) denotes the five-year delta of the swap portfolio and \(\Delta_{5Y}^{\text{par}}\) denotes the five-year delta of a five-year par swap. The sign of the equivalent position denotes which type of swap, receiver or payer, should be entered to offset the interest rate exposure. The expression \(\frac{\Delta_{5Y}^\pi}{\Delta_{5Y}^{\text{par}}}\) is often referred to as the hedge ratio, denoted \(h_j\), and a portfolio of equivalent positions is called a hedging portfolio. Following Bianchetti (2010), the hedge ratio for a specific point \(j\) (for example the five-year point as in the example above), can more generally be defined as

\[
h_j = \frac{\partial \Delta_j^\pi}{\partial \delta_j^H}
\]  

(6.3)

where \(h_j\) is the hedge ratio for point \(j\), \(\Delta_j^\pi\) is the delta of the portfolio for point \(j\) and \(\delta_j^H\) is the delta of the hedging instrument, \(H\), for a specific point, \(j\). The quantity \(\Delta_j^\pi\) used in Equation (6.3) is defined as

\[
\Delta_j^\pi = \frac{\partial \text{PV}^\pi}{\partial L_j}
\]  

(6.4)

where \(\text{PV}^\pi\) denotes the present value of the portfolio and \(L_j\) denotes the market quote for a specific maturity \(T_j\). The quantity \(\delta_j^H\) in Equation (6.3) is defined as
where $PV_j^H$ is the present value of the hedging instrument $H$ for point $j$. The hedging instruments are the same as the instruments used to build the curve, or alternatively only a few of those instruments are used in the hedging.

Following the approach of Bianchetti (2010), the total value of the position (swap portfolio plus hedging portfolio) is defined as

$$PV_{\text{Total}}^\pi = PV^\pi - \sum_{j=1}^{N_H} h_j PV_j^H$$

(6.6)

where $PV^\pi$ denotes the value of the swap position, $N_H$ represents the number of hedging instruments, $h_j$ is the hedge ratio for point $j$, and $PV_j^H$ denotes the value of the hedging instrument $H$ for point $j$. To be perfectly hedged, the hedging portfolio is constructed to exactly offset the changes in the swap portfolio. That is, if Equation (6.6) is rewritten using the deltas of the positions, the total delta will be

$$\Delta_{\text{Total}}^\pi \approx \sum_{k=1}^{N_H} \left[ \frac{\partial PV^\pi}{\partial L_k} - \sum_{j=1}^{N_H} h_j \frac{\partial PV_j^H}{\partial L_j} \right]$$

$$= \sum_{k=1}^{N_H} \left[ \frac{\partial PV^\pi}{\partial L_k} - h_k \delta_k^H \right] = 0.$$  

(6.7)

From Equation (6.7) we see that the total position, the original swap portfolio plus the hedging portfolio, has zero delta. This, in turn, implies that the portfolio is delta neutral, i.e. completely hedged against small changes in interest rates. However, as time passes, the delta will change and the hedging portfolio needs to be rebalanced to maintain a zero delta. When the hedging portfolio is rebalanced frequently it is called to hedge dynamically, whereas static hedging is when the hedge ratios are computed once only. The delta hedge only protects against small changes in interest rates. To hedge large changes one also needs to consider the gamma measure which is the second derivative, i.e. the derivative of the delta, with respect to the underlying. Gamma hedging is out of the scope of this thesis, but the interested reader can find more information about gamma in for example Flavell (2010, p.284 ff).
6.3. Hedging in the multiple-curve framework

In the multiple-curve framework, different curves are used for deriving forward rates and discount factors. Since several curves are used in the pricing there will also be two or more deltas; one delta (or several if we have a portfolio of interest rate derivatives with different underlying tenors) that relates to changes in the spot curve, which is used as an input in the derivation of the forward curve, and one delta that relates to changes in the Eonia OIS rates that make up the discount curve. These deltas are denoted $\Delta_{\text{Fwd curve}}^\pi$ and $\Delta_{\text{Disc curve}}^\pi$ respectively. Further, according to Henrard (2009), if the OIS discount curve has been used in the construction of the forward curve there is also a need to consider an indirect discounting-delta. Out of these different deltas the direct forward-curve-delta $\Delta_{\text{Fwd curve}}^\pi$ is by far the most important. Henrard (2009) finds that when calculating deltas for different fixed-rate swaps, the two discount-deltas correspond to at the most seven percent of the total delta.

As in the single-curve approach, the delta corresponding to the forward curve is found by separately increasing the instruments included in the forward curve by one basis point and measuring the change in the present value of the swap. The delta corresponding to the discounting curve is found by doing the same thing with the instruments included in the OIS-discount curve. Following the approach of Bianchetti (2010), the total delta is thus defined as

$$
\Delta_{\text{Total}}^\pi = \sum_{k=1}^{N_C} \sum_{j=1}^{N_k^L} \frac{\partial \text{PV}^\pi}{\partial L_{kj}}
$$

(6.8)

where $N_C$ is the number of curves (two or more), $N_k^L$ denotes the number of bootstrapping instruments (for all curves), and $L_{kj}$ denotes the market quotes of these instruments.

As mentioned above, the delta referring to the discount curve and the delta referring to the forward curve differ. Following Bianchetti (2010), the delta referring to the discount curve $\Delta_{\text{Disc curve}}^\pi$ can be expressed as

$$
\Delta_{\text{Disc curve}}^\pi = \sum_{j=1}^{N_d^R} \sum_{a=1}^{N_a^R} \frac{\partial \text{PV}^\pi}{\partial R(t,T_{a}^d)} \frac{\partial R(t,T_{a}^d)}{\partial L_{d,j}}
$$

(6.9)

where $L_{d,j}$ denotes the Eonia OIS market quotes for different maturities and $R(t,T_{a}^d)$ denotes a vector of $N_{d}^R$ spot rates, that is, spot rates from where the discount curve is derived. A change
in the market quotes affects the derived OIS spot rates which in turn affects the present value of the swap position. A similar delta can be derived in relation to the forward curve $\Delta_{\text{Fwd curve}}^\pi$ defined as

$$
\Delta_{\text{Fwd curve}}^\pi = \sum_{j=1}^{N_f} \sum_{a=1}^{N_f^R} \frac{\partial PV^\pi}{\partial R(t, T_{d}^f)} \frac{\partial R(t, T_{d}^f)}{\partial L_{r,j}}
$$

(6.10)

where $L_{r,j}$ is the market quotes for the instruments used to bootstrap the forward curve and $R(t, T_{d}^f)$ is a vector of $N_f^R$ spot rates that relates to the forward curve. Again, the present value of the swap position is altered by a change in the spot rates, which in turn depends on the market quotes. If OIS discount factors are used, together with Euribor market quotes, to build the forward curve (as in Henrard, 2009) there will also be an indirect delta. Bianchetti (2010) describes the indirect effect on the forward delta as an additional term in Equation (6.10), that is

$$
\Delta_{\text{Fwd curve}}^\pi = \sum_{j=1}^{N_f} \sum_{a=1}^{N_f^R} \frac{\partial PV^\pi}{\partial R(t, T_{d}^f)} \frac{\partial R(t, T_{d}^f)}{\partial L_{r,j}} + \sum_{j=1}^{N_d} \sum_{a=1}^{N_f^R} \frac{\partial PV^\pi}{\partial R(t, T_{d}^f)} \frac{\partial R(t, T_{d}^f)}{\partial L_{d,j}}
$$

(6.11)

where $L_{d,j}$ denotes the Eonia OIS market quotes. The spot rate referring to the forward curve can thus change in two ways, by a change in the Euribor market quotes $L_{r,j}$ (the direct effect) or by a change in the Eonia OIS market quotes $L_{d,j}$ (the indirect effect). We repeat however that the indirect effect will only matter if the OIS discount factors is used in the forward curve building process.

The risk associated with changes in the Euribor spot curve, from where the forward curve is derived, is hedged using swaps (or swaps and forward rate agreements) whereas the risk associated with changes in the OIS curve (both direct and indirect risk) is hedged using overnight index swaps (Eonia OIS). The hedge ratios are calculated as in the single-curve approach (see Equation (6.3)).

In the following section we perform some of the theory addressed in this section in practice.
7. Comparison of the pre-crisis and post-crisis frameworks

In this section, we compare the single-curve and multiple-curve frameworks in practice. First, in Subsection 7.1, we compare the discount curve, forward curve and spot curve in the two approaches. Then, in Subsection 7.2 we calculate par swap rates for an interest rate swap, measure the delta sensitivity and compute the hedge ratios.

The main differences between the single-curve and multiple-curve frameworks relates to how the forward curves and discount curves are constructed. This in turn has implications for the delta hedging, as described in Section 6 above, since an interest rate swap in the multiple-curve framework is exposed to different types of delta risk. In the subsections below we will compare the two pricing frameworks in a practical manner by considering the following scenario;

Suppose that we sell a 3.5-year interest rate swap with a notional amount of 10 000 000 euro. That is, for the next 3.5 years we will make semi-annual floating rate payments and in exchange receive annual fixed rate payments. If interest rates decrease, the swap increases in value since it means that we pay less but still receive the same fixed rate. Naturally, if interest rates increase, our position decreases in value. The risk associated with the swap is thus increasing interest rates. How can this swap be priced and hedged under the two different frameworks that have been addressed in Sections 3 and 5?

We show all results for two points in time; November 30, 2010 and November 30, 2011. The two points are indicated with circles in Figure 7.1. The dates are chosen with respect to the Euribor-Eonia OIS spread where the later date displays a greater spread than the former (because of the sovereign debt crisis of 2011). The aim is to see whether or not the size of the spreads in the market affects the difference between the two frameworks. Further, the reason why we do not choose September 2008 as one of the dates, even though the greatest spreads were observed at that time, is simply due to the fact that the Eonia OIS data for longer maturities is limited before mid-2009.
The two points in time that are examined in the numerical part of the thesis are indicated with circles. The spread is considerable larger in November 2011 than in November 2010.

7.1. **Comparison of curves**

In the single-curve framework, the discount curve is derived from Euribor rates whereas in the multiple-curve framework, Eonia swap rates are used. A comparison of the Eonia OIS discount curve and the Euribor discount curve is displayed in Figure 7.2. The data is for the two points in time mentioned above; November 30, 2010 and November 30, 2011. As can be seen in the figure, the Eonia OIS discount factors in the multiple-curve approach are greater than the Euribor discount factors in the single-curve approach. This difference is due to the fact that Eonia OIS quotes are lower than Euribor quotes as a consequence of the different credit and liquidity risk inherent in the rates (as discussed in Subsection 4.4). Further, as the interest rates have decreased between the two dates the discount curves have shifted to the right.
Figure 7.2. Comparison of the discount curves between the single-curve and multiple-curve frameworks. The discount curve in the multiple-curve framework is derived from the Eonia OIS rates. Since Eonia OIS rates are considered to be risk-free they are quoted lower than Euribor rates, consequently leading to larger discount factors. The comparison is for two points in time; November 30, 2010 and November 30, 2011.

Although the discount curves in the two frameworks are constructed using completely different instruments, their forward curves are computed in a similar way and therefore barely differ. Figure 7.3 displays a comparison between the forward curves in the two frameworks. The multiple-curve framework seems to provide a somewhat smoother curve and the main difference can be observed for maturities less than two years. However, there is no clear pattern of whether the multiple-curve forward curve should be below or above the single-curve equivalent. Instead it is above at very short maturities (less than a year) and below in mid-term maturities. The long ends, though, roughly coincide due to the use of the same bootstrapping instruments. The forward curve appears to have changed shape between November 2010 and November 2011 since the forward rates for very long maturities (25-30 years) are quoted higher whereas the forward rates for other maturities have decreased.
Source: Nordea Analytics/Bloomberg and own computations

**Figure 7.3. Comparison of the forward curves between the single-curve and multiple-curve frameworks.** Roughly the same bootstrapping instruments are used in the construction of the forward curves in the single-curve and multiple-curve frameworks, which leads to rather similar curves. The observations are for November 30, 2010 and November 30, 2011.

Another fundamental curve, in addition to the discount curve and the forward curve, is the spot curve. A comparison of the spot curves under the single-curve and multiple-curve frameworks for two points in time (November 30, 2010 and November 20, 2011 as before) is presented in Figure 7.4. However, in fact there are three types of spot curves where the third spot curve is the Eonia OIS spot curve related to the Eonia OIS discount curve. As could be expected, the Eonia OIS spot curves (the dot-dashed lines) are below the other curves (recall that Eonia rates are lower due to a lower credit risk) and differences between the two Euribor curves are found for maturities of less than two years, where the multiple-curve approach shows higher spot rates than the single-curve approach. This divergence is a result of the different interest rate tenors, since the mid-end of the single-curve spot curve is constructed using interest rate tenors of three months whereas a tenor of six months is used in the multiple-curve framework due to the post-crisis requirement of homogeneity (all bootstrapping instruments should have the same underlying interest rate tenor).

From Figure 7.4, it is also clear that there has been a shift downwards in the spot curve between the two dates. The shift is not completely parallel but varies between zero and approximately 50 basis points (0.5%). This is important later, in Subsection 7.2, when we
delta-hedge our swap in order to protect the position from these types of changes in interest rates.

![Comparison of the spot curves between the single-curve and multiple-curve frameworks.](image)

**Source:** Nordea Analytics/Bloomberg and own computations

**Figure 7.4. Comparison of the spot curves between the single-curve and multiple-curve frameworks.** The multiple-curve approach has two spot curves, one related to the forward curve and another related to the Eonia OIS discount curve. Eonia OIS quotes are lower than Euribor quotes (due to lower credit risk) and consequently the Eonia OIS spot curve is below the other two. Regarding the spot curve related to the forward curve, the only difference between the frameworks can be seen for maturities of less than two years since this is the only interval where different bootstrapping instruments are used. The spot curves are plotted for November 30, 2010 and November 30, 2011.

How do we know if these curves are suitable for pricing interest rate swaps? The level of smoothness can be a good indicator. Here, smoothness means that the spot curve is represented by a function with a continuous first derivative (Adams, 2001). Adams (2001) argues that a smooth spot curve provides more accurate pricing of securities. Since the discount factors as well as the spot rates are obtained through the integration of the instantaneous forward rate (recall Figure 3.1 in Section 3), the spot and discount curves are, by construction, smoother than the forwarding curve. The forward curve will therefore be a good indicator when evaluating the spot curve (Ametrano and Bianchetti, 2009). We conclude that the forward curves in Figure 7.3 are fairly smooth.
To examine the validity of the derived curves we can also compare them to curves constructed in related research. Bianchetti and Carlicchi (2011) plot the Euribor standard spot curve (i.e. with mixed underlying interest rate tenors), the 6M Euribor spot curve (6M Euribor as underlying) and the Eonia OIS spot curve in the same figure. A small difference can be seen for maturities of less than three years where the standard Euribor curve is somewhat below the 6M Euribor curve. Further, the Eonia OIS curve is consistently below the other curves. This is in line with our results in Figure 7.4. Furthermore, the forward curves in Figure 7.3 are similar to those derived in the paper of Bianchetti and Carlicchi (2011) who find the post-crisis forward curve to be smoother, especially for mid-term maturities, than the pre-crisis forward curve.

7.2. Pricing and hedging

The delta-hedging of interest rate swaps was addressed in Section 6. In this subsection we derive hedge ratios for a position in a 3.5-year swap using data from the Euro swap market. Since 3.5-year swaps are not quoted on the market, the derived curves that were shown in Subsection 7.1 above are applied to calculate the par swap rate (the fixed rate that makes the present values of both legs in the swap agreement equal). Then we determine the delta sensitivity, that is, the sensitivity to small changes in interest rates, by calculating the change in the present value after small “bumps” to each of the rates that make up the pricing curves. Finally we calculate the hedge ratios, defined in Equation (6.3), in order to create a hedging portfolio that offsets this risk.

The swap agreement in question is described in Table 7.1. We make the computations for two points in time; November 30, 2010 and November 30, 2011. As mentioned in the beginning of Section 7, these dates are characterized by different Euribor-Eonia OIS spreads. For simplicity we assume that both legs are paid semi-annually.
The swap agreement has a floating leg indexed to 6M Euribor and a notional amount of EUR 10 000 000. We consider two different start dates; November 30, 2010 and November 30, 2011.

The 3.5-year swap is not quoted on the market. The first step is therefore to calculate the par swap rate. The present values of the legs are found by summing up the discounted cash flows as described in Section 3 and 5 respectively. In the previous subsection, we showed that the forward and discount curves differ both from one point in time to another as well as with respect to the pricing framework, i.e. the single-curve or the multiple-curve framework. Consequently we find four different swap rates that each makes the present value of the swap equal to zero at initiation $t_0$. The calculated par rates, together with the difference (measured as single-curve par rates minus multiple-curve par rates) are summarized in Table 7.2.

<table>
<thead>
<tr>
<th>2010-11-30</th>
<th>2011-11-30</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single-curve</td>
<td>0.018843</td>
</tr>
<tr>
<td>Multiple-curve</td>
<td>0.018882</td>
</tr>
<tr>
<td>Difference (bps)</td>
<td>-0.393396</td>
</tr>
</tbody>
</table>

Table 7.2. Calculated par-swap rates for a 3.5-year swap. The swap rates are calculated for two start dates using the different pricing frameworks. The difference between the frameworks is larger in November 2011 than in November 2010.

As can be seen in Table 7.2, the calculated par swap rates for the 3.5-year swap differ somewhat between the two pricing frameworks. In the first case, the difference is 0.39 basis points whereas in the second case, the difference is 0.60 basis points. These may seem like small numbers but since a swap agreement usually has a notional of several millions and since financial institutions usually have a lot of swaps in their portfolio, these differences are in fact important.

Numerical examples of the difference between the pre- and post-crisis approaches are scarce. However, Whittall (2010) is quoting a trader at Barclays Capital that estimates the difference
to be “up to 65 basis points on the swap rate, depending on the exact tenor of the swap” (Whittall, 2010, p.3). It is also said that the single-curve swap rate is lower than the multiple-curve swap rate, which is something that has made banks keep the single-curve approach to get a comparative advantage (and lure customers with lower swap rates) (Whittall, 2010).

Next, we measure how sensitive our swap is to changes in interest rates, which is done by increasing each instrument underlying the spot curve with one basis point (0.01%) and then calculating the change in present value. In the multiple-curve case, this change is done both to interest rates underlying the forward curve and to interest rates used to bootstrap the discount curve. Consequently, we get two types of deltas. The indirect delta (discussed in Subsection 6.3) cause no concern in our case since we are not using Eonia OIS discount factors in the computation of the forward curve, as motivated in Section 5. Again, we make the computations for two points in time. The results are collected in a so called delta vector, which was discussed in Section 6. The delta vectors can be seen in Table 7.3. Note that we have left some small exposures out for convenience.

<table>
<thead>
<tr>
<th></th>
<th>2010-11-30</th>
<th>2011-11-30</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Single-curve</td>
<td>Multiple-curve</td>
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<tr>
<td>2Y</td>
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<td>324.73</td>
</tr>
<tr>
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<td>-1,804.00</td>
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<td>4Y</td>
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<tr>
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</tr>
<tr>
<td><strong>Total Delta</strong></td>
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<td>-3,342.00</td>
</tr>
<tr>
<td>• Forward delta</td>
<td>-3,358.40</td>
<td>-3,384.60</td>
</tr>
<tr>
<td>• Discount delta</td>
<td>16.37</td>
<td>9.23</td>
</tr>
</tbody>
</table>

Table 7.3. Delta vectors for a 3.5-year swap. The multiple-curve framework shows consistently larger deltas for both points in time. In the multiple-curve framework, the forward delta dominates the discount delta.

As proposed by Henrard (2009), the delta related to the forward curve is by far the most important delta in the multiple-curve approach. He finds the discount delta to account for between -7% and +7% of the total delta depending on how far the swap is from par. According to Henrard (2009), a swap that is at par should have zero discount delta (including both direct and indirect delta). In our example, the discount delta accounts for approximately -0.5% when spreads are low and -0.3% when spreads are high. In other words, the discount delta is close to zero.
In general, the multiple-curve case shows larger deltas than the single-curve case. This difference is slightly more visible in November, 2011 than in November, 2010. Consequently, if the multiple-curve is the “right” framework to use, as most market participants agree on (Cameron, 2011), this means that the swap traders that continue to use the old, single-curve framework underestimate the risk that their swap positions are exposed to.

As can be seen in Table 7.3, the swap is most exposed to changes in three-year and four-year interest rates, which is true for both frameworks. Hence, we decide to focus on hedging these sensitivities. The hedge ratios are found by taking the ratio of the delta of the swap portfolio and the delta of the hedging instrument. We use three-year and four-year par swaps as hedging instruments. The delta sensitivities of the swap and the following hedge ratios are summarized in Table 7.4.

<table>
<thead>
<tr>
<th></th>
<th>2010-11-30</th>
<th>2011-11-30</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Single-curve</td>
<td>Multiple-curve</td>
</tr>
<tr>
<td>Delta 3-year par swap</td>
<td>-2,861.00</td>
<td>-2,914.80</td>
</tr>
<tr>
<td>Delta 4-year par swap</td>
<td>-3,756.10</td>
<td>-3,852.60</td>
</tr>
<tr>
<td>3-year hedge ratio</td>
<td>0.607</td>
<td>0.619</td>
</tr>
<tr>
<td>4-year hedge ratio</td>
<td>0.604</td>
<td>0.600</td>
</tr>
</tbody>
</table>

**Table 7.4. Comparison of hedge ratios.** The hedge ratios are found by dividing the portfolio’s delta sensitivity by the delta sensitivity of the hedging instrument. The ratios are similar between the frameworks but the multiple-curve approach shows somewhat larger hedge ratios.

The differences in hedge ratios between the frameworks are small and no difference can be seen between the case of low spreads (November, 2010) and the case of high spreads (November, 2011). The three-year hedge ratio is approximately the same between frameworks whereas the four-year hedge ratio differs somewhat. The positive sign of the ratios indicate that we shall enter a payer interest rate swap to offset the exposure of our receiver interest rate swap.

To offset the risk that our position is exposed to we multiply the hedge ratios by the notional amount of the swap we want to hedge, i.e. 10 million. Then we buy payer interest rate swaps where the notional corresponds to the calculated amount. Even though the differences between the hedge ratios in Table 7.4 appeared to be very small, there is a considerable difference in notional amount as can be seen in Figure 7.5.
Figure 7.5. Comparison of equivalent positions between the single-curve and multiple-curve frameworks. The equivalent position, which is the position that one can take in order to offset the interest rate risk exposure, is found by multiplying the notional amount with the hedge ratio.

Once again, we want to stress that the market for interest rate derivatives are extremely large and that a swap portfolio may be worth several million euro. Even the smallest miscalculation can therefore lead to big losses. It is thus important to use the correct pricing framework, i.e. the multiple-curve framework.

8. Conclusion

The aim of this thesis was to describe the evolvement of interest rate swap (IRS) pricing and how the financial crisis of 2007-2009 caused a regime change in interest rate markets, and brought about the transition from a single-curve pricing approach to a multiple-curve pricing approach. The interest rate swap market has grown rapidly since its initiation in the early 1980s and interest rate swaps are frequently used by financial institutions and large corporations for risk management purposes. It is therefore highly important to be able to correctly price this instrument.

Prior to the crisis, IRS pricing was considered to be straightforward and relied on the assumption that borrowing and lending could take place at a unique risk-free rate. Forward rates, cash flows, and discount factors could all be computed on the same curve, thereby the
name single-curve approach. In mid-2007, however, at the onset of the financial crisis, the market environment radically changed and rates that followed each other closely prior to the crisis started to diverge and displayed non-negligible spreads. Researchers and market participants agree that the abnormalities observed on the interest rate markets since August 2007 to a large extent can be explained by credit and liquidity issues. Euribor can, for example, no longer be considered a reliable risk-free proxy, since Euribor rates with different underlying tenors exhibit different credit and liquidity risks. Thus, the single-curve pricing approach using Euribor for discounting has been abandoned.

The best risk-free proxy in the post-crisis environment has turned out to be the overnight indexed swap (OIS) rate, which is the Eonia rate in the Euro market, and so discounting should be done using the OIS discount curve. There is also a need for multiple forwarding curves with homogenous underlying tenors, which is why the post-crisis pricing approach has come to be called the multiple-curve framework.

The second purpose of this thesis was to compare the pre- and post-crisis frameworks empirically by pricing and hedging an interest rate swap under both approaches. This was done by deriving the single spot curve, used in the pre-crisis framework, and the homogenous forward curve and the OIS discounting curve, used in the post-crisis multiple-curve framework. The curves were then applied in the computation of the par swap rate of a 3.5-year interest rate swap for two different start dates; November 30, 2010 and November 30, 2011. The start dates were chosen with respect to the observed Euribor-Eonia OIS spread where the latter date show a considerably larger spread due to the ongoing sovereign debt crisis. Finally, delta sensitivities and hedge ratios were calculated and compared. We found that the multiple-curve framework computed higher par rates and delta sensitivities. A somewhat greater difference could be observed when large Euribor-Eonia OIS spreads were present in the market than under more normal market conditions. The results imply that the pre-crisis single-curve framework underestimates the price of the swap as well as the delta risk that the swap position is exposed to. Hence, it is crucial to adopt the post-crisis multiple-curve framework, which is considered to be the “right” pricing approach today.
References


Bianchetti, M., 2008, ”Two curves, one price: Pricing and hedging interest rate derivatives using different yield curves for discounting and forwarding”.


Capponi, A., 2011, “Pricing and mitigation of counterparty credit exposures”.


Filipović, D., Trolle, A.B., 2011, ”The term structure of interbank risk”.


Herbertsson, A., 2011, “Credit risk modeling: Lecture notes”.


Electronic sources


Appendix A

Reference rates
Libor is the interbank reference rate between London based banks. The rate is quoted for ten different currencies with maturities ranging from one day to twelve months. Each day, a panel of London banks (7-18 banks depending on the currency) submits their cost of borrowing unsecured funds, i.e. the rate at which the banks can obtain funding, without depositing collateral, for some maturity in a specific currency on the London interbank money market (bbalibor.com). In the Euro market, the Libor equivalent is Euribor. The panel of banks, currently 44 banks, contributing to Euribor consists of the banks with the highest volume of business in the euro zone money markets (euribor-ebf.eu). Euribor is quoted for maturities ranging from one week to twelve months. The reference rate for overnight lending in the euro zone is the Eonia (Euro overnight index average) which is obtained in the same way as the Euribor rates.

Appendix B

Data used for construction of the short end of the spot curve in the single-curve framework.

<table>
<thead>
<tr>
<th>Date</th>
<th>30-11-2010</th>
<th>30-11-2011</th>
</tr>
</thead>
<tbody>
<tr>
<td>EONIA O/N</td>
<td>0.54</td>
<td>0.81</td>
</tr>
<tr>
<td>EURIBOR 1W</td>
<td>0.60</td>
<td>0.90</td>
</tr>
<tr>
<td>EURIBOR 2W</td>
<td>0.66</td>
<td>0.99</td>
</tr>
<tr>
<td>EURIBOR 1M</td>
<td>0.81</td>
<td>1.21</td>
</tr>
<tr>
<td>EURIBOR 2M</td>
<td>0.91</td>
<td>1.31</td>
</tr>
<tr>
<td>EURIBOR 3M</td>
<td>1.03</td>
<td>1.47</td>
</tr>
</tbody>
</table>

Source: Nordea Analytics/Bloomberg

Market quotes in percent for Eonia and three Euribor deposit rates.
Data used for construction of the middle area of the spot curve in the single-curve approach.

<table>
<thead>
<tr>
<th>Date</th>
<th>2010-11-30</th>
<th>2011-11-30</th>
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<tbody>
<tr>
<td>EUR FRA 1x4</td>
<td>1.06</td>
<td>1.21</td>
</tr>
<tr>
<td>EUR FRA 2x5</td>
<td>1.12</td>
<td>1.07</td>
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<tr>
<td>EUR FRA 3x6</td>
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<td>1.06</td>
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<tr>
<td>EUR FRA 4x7</td>
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<td>EUR FRA 5x8</td>
<td>1.20</td>
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<tr>
<td>EUR FRA 6x9</td>
<td>1.22</td>
<td>1.01</td>
</tr>
<tr>
<td>EUR FRA 7x10</td>
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<td>0.96</td>
</tr>
<tr>
<td>EUR FRA 8x11</td>
<td>1.26</td>
<td>0.96</td>
</tr>
<tr>
<td>EUR FRA 9x12</td>
<td>1.29</td>
<td>1.02</td>
</tr>
</tbody>
</table>

*Source: Nordea Analytics/Bloomberg*

Market quotes in percent for nine different FRAs.

Data used for the construction of the long end of the spot curve in both the single-curve and multiple-curve framework.

<table>
<thead>
<tr>
<th>Date</th>
<th>2010-11-30</th>
<th>2011-11-30</th>
</tr>
</thead>
<tbody>
<tr>
<td>EUR SWAP 2Y</td>
<td>1.55</td>
<td>1.41</td>
</tr>
<tr>
<td>EUR SWAP 3Y</td>
<td>1.78</td>
<td>1.56</td>
</tr>
<tr>
<td>EUR SWAP 4Y</td>
<td>2.04</td>
<td>1.79</td>
</tr>
<tr>
<td>EUR SWAP 5Y</td>
<td>2.29</td>
<td>2.02</td>
</tr>
<tr>
<td>EUR SWAP 6Y</td>
<td>2.50</td>
<td>2.22</td>
</tr>
<tr>
<td>EUR SWAP 7Y</td>
<td>2.67</td>
<td>2.38</td>
</tr>
<tr>
<td>EUR SWAP 8Y</td>
<td>2.81</td>
<td>2.51</td>
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<tr>
<td>EUR SWAP 9Y</td>
<td>2.93</td>
<td>2.62</td>
</tr>
<tr>
<td>EUR SWAP 10Y</td>
<td>3.03</td>
<td>2.72</td>
</tr>
<tr>
<td>EUR SWAP 12Y</td>
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<td>2.88</td>
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<td>EUR SWAP 15Y</td>
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<td>3.02</td>
</tr>
<tr>
<td>EUR SWAP 20Y</td>
<td>3.41</td>
<td>3.03</td>
</tr>
<tr>
<td>EUR SWAP 25Y</td>
<td>3.31</td>
<td>2.94</td>
</tr>
<tr>
<td>EUR SWAP 30Y</td>
<td>3.16</td>
<td>2.85</td>
</tr>
</tbody>
</table>

*Source: Nordea Analytics/Bloomberg*

Market quotes in percent for Euro swaps ranging from two years to 30 years.
Appendix C

Comparison of different interpolation schemes.

Appendix D

Data for the construction of the short end and middle area in the multiple-curve approach.

<table>
<thead>
<tr>
<th>Date</th>
<th>30-11-2010</th>
<th>30-11-2011</th>
</tr>
</thead>
<tbody>
<tr>
<td>EONIA O/N</td>
<td>0.54</td>
<td>0.81</td>
</tr>
<tr>
<td>EURIBOR 6M</td>
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<td>1.70</td>
</tr>
<tr>
<td>EUR FRA 1x7</td>
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<td>1.43</td>
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<tr>
<td>EUR FRA 2x8</td>
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<td>1.36</td>
</tr>
<tr>
<td>EUR FRA 3x9</td>
<td>1.35</td>
<td>1.35</td>
</tr>
<tr>
<td>EUR FRA 4x10</td>
<td>1.37</td>
<td>1.26</td>
</tr>
<tr>
<td>EUR FRA 5x11</td>
<td>1.39</td>
<td>1.23</td>
</tr>
<tr>
<td>EUR FRA 6x12</td>
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<td>1.25</td>
</tr>
<tr>
<td>EUR FRA 12x18</td>
<td>1.61</td>
<td>1.25</td>
</tr>
</tbody>
</table>

Source: Nordea Analytics/Bloomberg

Market quotes in percent.